ON THE SYSTEM OF INTERREGIONAL COMMODITY FLOWS

TAKEO IHARA

I

The purpose of this paper is to show how appropriate models of spatial systems can be derived by maximizing a function describing the entropy or information contained in such systems subject to relevant constraints. This approach is specifically related to the concept of interaction in the system of interregional commodity flows. And it can be shown that the models which can be derived using entropy-maximizing methods are equivalent to many of the models already in use which have been derived empirically. Accordingly, in the process of this review, theory-building and verification are the main concerns.

This paper begins with the brief discussion on the input-output model. This section shows how to locate the input-output model within the framework of the general equilibrium theory, and also refers to the still remained areas about it. In Section 3, the Leontief-Strout multiregional model is examined as one of the extensive expansions of the input-output model. Section 4

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digs into the entropy-maximizing methods from the viewpoint of the model-building. It also evaluates the entropy-maixmizing model for the original Leontief-Strout version. The paper closes with some concluding remarks and a forward look.

II

The input-output model is known as one of the central subjects in the field of modern economics. For the purpose of simplicity, let us consider the *single-region, static, open input-output model*. The input-output analysis consists of the following three tables :

1) the transaction matrix table ;
2) the input coefficient matrix table ;
3) the inverse matrix table.

Among them, the first table is the most important. It has a property of double-entry system where every cell stands for an input as well as an output. Owing to this property, we come to obtain a clear idea of the structural characteristics of one industry compared with the others.

However, in order to make use of the transaction matrix table not only as the *descriptive device* but also as the *analytical tool*, we usually assume the following technical assumptions :

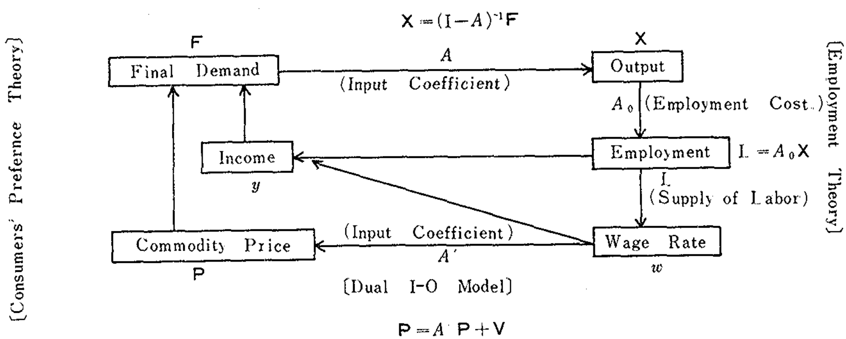
1) constant returns to scale ;
2) convexity of the isoquant surfaces ;
3) fixed coefficients of production.

If we admit all of these assumptions, the latter two tables (i. e., the input coefficient matrix table and the inverse matrix table) can be readily calculated to serve as efficient tools in a variety of economic problems. Therefore, whenever we are interested in the application of the input-output model, these assumptions should be theoretically as well as statistically tested.[1]

1) For the detailed discussion, see Ihara (2).

As previously stated, the development of the input-output model stems mainly from economics. Thus, for the purpose of better understanding the effective range and deficiencies of the input-output model, let us locate single-region, static, open input-output model within the *general equilibrium framework*. It is schematically pictured in Figure 1.

Figure 1
Diagrammatic Representation of the Input-Output Model[2)]
〔Input-Output Model〕



Consider the dual problem of the input-output model. Let w be an equilibrium price of labor, and let V be the column vector of the average value-added, which is defined as (wage payment per unit of each output) + (average profit per unit of each output). As a result of competition, the equality between price and cost holds with respect to each commodity. Hence, if we put $V = wA'_0$, which in turn determines the price vector P as follows :

$$P = (I - A')^{-1}\, wA'_0 = \{(I - A)^{-1}\}' wA'_0.$$

From this, we conclude with the following statements :

2) In this diagrammatic representation, the writer draws upon Morishima (7), and Dorfman, Samuelson, and Solow (1). In this Figure, note that the input coefficient matrix A is measured in physical units. Furthermore, if we replace "Employment" by the primary factors of production, and " Supply of Labor" and "Wage Rate" by the supplies and prices of the primary factors of production, respectively, we can view this chart in a more general way.

1) The input-output model takes a role of one part of the general equilibrium theory. Hence, it may be called a *partial theory*. It should be noted that the general equilibrium theory has a sophisticated mathematical structure, while the input-output model is rather operational, macrostate description.

2) From the viewpoint of the general equilibrium theory, the input-output model can be viewed to have a peculiar characteristic, and hence it is not always effective. Specifically, any price change induces the change in input coefficients.[3] However, in the input-output model, *the price-determining mechanism is independent of the output-determining*. This is one of the theoretical characteristics built in the input-output model.

Consider the separability conditions. If we assume the labor market of the Keynesian type[4] for example, then we can assert that the input coefficients remain unchanged in a case where the involuntary unemployments exist. As the result, the input-output relation tends to be stable. The Keynesian model can also be built in the input-output framework as follows :

From this we conclude that :

1) The Keynesian model deals with the aggregate values. Hence, it may be called the *aggregate theory*.

2) The input-output model and the Keynesian model are *supple-*

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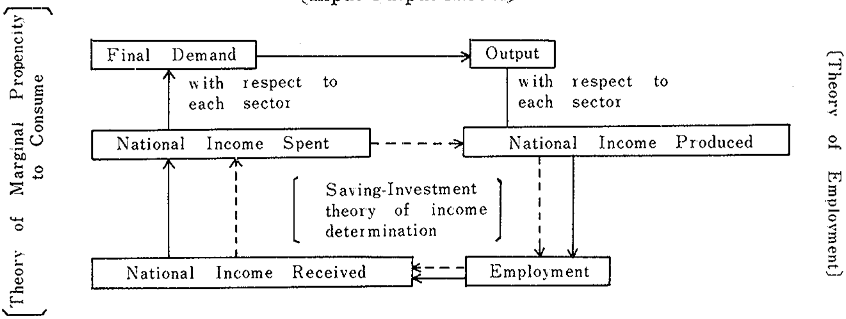
3) In a general case, any change in w causes the changes in P, which in turn cause the change in A and A_0. In other words, the relative price changes are not allowed to take place in the conventional input-output model.

4) The labor market of the Keynesian type can be written as follows ; $y=y(n)$ … production function, $w=y'(n)$ … real wage rate equals the marginal productivity of labor, $W=W_0+W(n)$ where $W=Pw$ … labor supply function. The Keynesian model has a labor supply function, which relates the labor offered to the *money wage* rate (instead of the real wage rate), and it introduces *an inflexible rate* W_0 for employment below a certain level $\bar{n}$. In a case of this inflexible money wage rate W_0, the excess supply of labor does not induce the decreasing change in wage rate, hence the relative price changes do not take place.

Figure 2
The Relationships between Keynesian and Input-Output Models
〔Input-Output Model〕



mentary one another, but not always so when viewed from the theoretical base and data check.

3) The results are only identical under the following special conditions ;

   i) Any relative change in National Income Produced (which is disaggregated at sector base) does not cause the change in the relative share of the National Income Recieved.

   ii) There are no significant differences in the propensity to consume for each class.

When we turn to the state of the interregional as well as intraregional economy, the situations seem to be more complicated. For this reason, a consistent and systematic way of approach is highly required to tackle the complicated situations. Although there are a few methods to grasp the economic structure, here we employ *the input-output model* among other methods, and *set it at the base* for our study below.

As for the orientation of how to develop the input-output model, we can suggest the following two big categories :

1) Intensive expansion oriented;

   To close the model with respect to the exogenous sectors (e.g., households) ; to convert the static model into the operational dynamic one contained the capital coefficients ; the impact analysis with an aid of the varied intersectoral multipliers, etc.

2) Extensive expansion oriented ;

To link the input-output model to the other kind of models, such as the econometric models, the linear programming model, the industrial complex analysis, the gravity model, etc.

The latter is the line which we now pursue.

III

As regional economic research has expanded in recent years, input-output has been used as the basic research tool for many of the regional studies. The economic analysis usually has been restricted to an isolated region, although some multiregional input-output studies have been completed. The latter generally are more difficult to implement, because the data requirements are greater. When we are concerned with interregional trade, information on the flows of goods and services among regions also must be assembled.

Spatially differentiated general equilibrium models have been used to estimate the interregional trade flows for aggregate commodity groups. Moses (1960) tested a linear programming model explaining shipments of all goods within the United States, but the empirical results were not very reasonable. The linear programming model generates implausible results, especially in cases where non-homogeneous products must be combined, since for composite products much cross-hauling (simultaneous flows of the same commodity between two regions) generally is observed.

More recently, a gravity trade model has been advocated by Leontief and Strout (1963) for use within a spatial, general equilibrium model, because it requires only a minimum of basic, factual information, and also permits the occurrence of cross-hauls among regions.

The gravity model was first discussed for use with regional input-output models by Isard (1960) as a possible means of estimating commodity shipments. In 1963, Leontief and Strout presented a form of the gravity trade model which can be readily implemented for a multiregional input-output analysis.[5)]

The Leontief-Strout Gravity Trade Model

The Leontief-Strout gravity trade model is specified by the following basic sets of equations :

$$(1)\quad x^m_{\cdot i} = \sum_n a^i_{mn}\, x^n_{i\cdot} + y^m_i$$

$$(2)\quad x^m_{i\cdot} = \sum_j x^m_{ij}$$

$$(3)\quad x^m_{\cdot j} = \sum_i x^m_{ij}$$

$$(4)\quad x^m_{ij} = \frac{x^n_{i\cdot}\quad x^m_{\cdot j}}{x^m_{\cdot\cdot}} \cdot q^m_{ij}$$

where $m, n = 1, 2, \ldots, p : i, j = 1, 2, \ldots, q,$

$q^m_{ii} = 0.$

The notation used in the equations includes :

- x^m_{ij} the amount of commodity m produced in region i which is shipped to region j,
- $x^m_{\cdot i}$ the total amount of commodity m demanded by all final and intermediate consumers in region i,
- $x^n_{i\cdot}$ the total amount of commodity n produced in region i,
- $x^m_{\cdot\cdot}$ the total amount of commodity m produced (consumed) in all regions,

5) Since this model combines the interindustry model and a gravity transportaion model, regional outputs and interregional shipments of commodities are determined simultaneously. However, a consistent set of regional input-output tables with interregional flows specified should be available. Poleenske (1970) describes the implementation of the complete model of this type in a multiregional input-output analysis of the Japanese economy.

y_i^m the total amount of commodity m demanded by final users in region i,

a_{mn}^i the amount of input of commodity m required by industry n located in region i to produce one unit of output of commodity n,

q_{ij}^m a trade parameter which is a function of the cost of transferring commodity m from region i to region j,

p the number of commodities,

q the number of regions.

The first equation shows that a balance exists between the total amount of commodity m demanded by the intermediate and final users within a region and the total amount supplied to that region. The second and third equations define the total production in region i and the total consumption in region j, respectively. The fourth equation states that the shipment of commodity m from region i to region j is proportional to the total production and total consumption of commodity m in the two regions respectively, and inversely proportional to the total amount of commodity m produced in all regions.[6)]

The above equation system permits simultaneous shipments of the same commodity to occur in both directions between two regions. In an actual economy, cross shipments of a product are often observed because data for commodity shipments are not assembled for strictly homogeneous products and are available on a regional, rather than a point-to point, and on an annual, rather than a monthly or a weekly, basis.

The multiregional system is completed by substituting the interregional trade eq. (4) first into eq. (2) and then into eq. (3) :

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6) Since the nonlinear interregional eq. (4) is homogeneous of degree one, proportional changes in regional outputs and supplies cause interregional shipments to vary by the same proportion.

$$(5)\quad x^m_{i.} = \sum_j x^m_{ij} = \frac{x^m_{i.} \sum_r (x^m_{.r}\, q^m_{ir})}{x^m_{..}} + x^m_{ii},$$

$$(6)\quad x^m_{.j} = \sum_i x^m_{ij} = \frac{x^m_{.j} \sum_r (x^m_{r.}\, q^m_{rj})}{x^m_{..}} + x^m_{jj},$$

where q^m_{ii}, $q^m_{jj} = 0$.

Eq. (6) can be rewritten with i substituted for j:

$$(6)'\quad x^m_{.i} = \frac{x^m_{.i} \sum_r (x^m_{r.}\, q^m_{ri})}{x^m_{..}} + x^m_{ii},$$

where $q^m_{ii} = 0$.

Eq. (5) shows that the production of commodity m in region i is equal to the total amount of commodity m produced and soled in the region plus the production sold to other regions. In a corresponding way, eq. (6) or eq. (6)′ indicates that total consumption of commodity m in region i is equal to the total amount of commodity m produced and used in the region plus the amount imported for consumption from other regions.

Assuming the final demands (y^m_i), the technical input coefficients (a^i_{mn}), and the trade parameters (q^m_{ij}) are known, the model is used to determine the total production of each commodity in region i ($x^m_{i.}$), the total consumption in region j ($x^m_{.j}$), and the amount of the commodity produced and used in region i (x^m_{ii}).

Figure 3

Relations between Equations and Unknowns

a) Equations			b) Unknowns		
(1)	… pq		$x^m_{i.}$	… pq	
(5)	… pq	$3pq$	$x^m_{.j}$	… pq	$3pq$
(6)	… pq		x^m_{ii}	… pq	

In order to implement the model, the basic system of equations is reworked into a simpler, more operational form. *The first step* is to reduce the number of equations and unknowns. *The second step* involves linearizing the structural interregional eqations.

Reduction of the number of variables

The reduction is accomplished by summing the two sets eqs. (2) and (3) above over all regions and subtracting one from the other. By this procedure, the pq variables x^m_{ii} can be eliminated.

Eq. (7) shows eqs. (2) and (3) summed over all regions :

(7) $\sum_i x^m_{i\cdot} = \sum_i \sum_j x^m_{ij} = \sum_j x^m_{\cdot j} = x^m_{\cdot\cdot}$,

where $m = 1, 2, \ldots, p$.

Eqs. (5) and (6)′ can be rewritten as :

(8) $x^m_{i\cdot}\, x^m_{\cdot\cdot} - x^m_{i\cdot} \sum_r (x^m_{\cdot r}\, q^m_{\cdot r}) = x^m_{\cdot i}\, x^m_{\cdot\cdot} - x^m_{\cdot i} \sum_r (x^m_{r\cdot}\, q^m_{ri})$,

where $m = 1, 2, \ldots, p$: $i = 1, 2, \ldots, q$: $\overset{m}{q}_{ii} = 0$.

Since the intraregional flows, the x^m_{ii}'s, have been eliminated, only $2pq$ variables remain to be determined by the $2pq$ equations. In fact, p of the equations are redundant in eq. (8) since any one of the q equations found by summing over regions in eq. (8) can be obtained from the other $q-1$ equations.

Since from eq. (7) the total supply of commodity m must equal the total demand for the commodity, additional p restrictions must be considered part of the system :

(9) $\sum_i x^m_{i\cdot} = \sum_i x^m_{\cdot i}$

where $m = 1, 2, \ldots, p$.

Eqs. (1), (8), and (9) constitute a system of $2pq$ equations in $2pq$ unknowns.

Figure 4
Relations between Equations and Unknowns

a) Equations

(1) … pq
(8) … pq (but p are redundant)
(9) … p

} $2pq$

b) Unknowns

$x^m_{i\cdot}$ … pq
$x^m_{\cdot j}$ … pq

} $2pq$

Linearization of the Model

The next process is to linearize the pq structural interregional eq. (8). The method used is to express all the endogenous and exogenousvariables as deviations from their base-year values:

(10) $x^m_{i\cdot} = \bar{x}^m_{i\cdot} + \triangle x^m_{i\cdot}$,

(11) $x^m_{\cdot i} = \bar{x}^m_{\cdot i} + \triangle x^m_{\cdot i}$,

(12) $y^m_i = \bar{y}^m_i + \triangle y^m_i$,

where $m = 1, 2, \ldots, p$; $i = 1, 2, \ldots, q$.

A barred variable represents the base-year value. All $\triangle$'s signify deviations from the base-year magnitude. From eqs. (5) and (6)′, we have eq. (8). Then, to obtain a linear approximation of eq. (8), we substitute in it eqs. (10) and (11).

In the resulting expression all terms containing a product of two barred letters will cancel out, because eq. (8) holds for the base year, and all the products of two deviations of variables can be dropped because they represent second-order terms. Thus the first-order approximation of eq. (8) takes the form of the following set of linear relationships:

(13) $\sum_r [\triangle x^m_{\cdot r} M^m_{ir}] - \sum_r [\triangle x^m_{r\cdot} N^m_{ri}] = 0,$

where $m = 1, 2, \ldots, p$: $i=1, 2, \ldots, (q-1)$.

The new constants M and N are introduced to simplify the form of these equations:

$$M^m_{ir} = \begin{cases} \bar{x}^m_{i\cdot} (1 - q^m_{ir}) & (\text{if } r \neq i) \\ \bar{x}^m_{i\cdot} - \bar{x}^m_{\cdot\cdot} + \sum_s (\bar{x}^m_{s\cdot} q^m_{si}) & (\text{if } r = i) \end{cases}$$

$$N^m_{ri} = \begin{cases} \bar{x}^m_{\cdot i} (1 - q^m_{ri}) & (\text{if } r \neq i) \\ \bar{x}^m_{\cdot i} - \bar{x}^m_{\cdot\cdot} + \sum_s (\bar{x}^m_{\cdot s} q^m_{is}) & (\text{if } r = i) \end{cases}$$

where $q^m_{ii} = 0$.

In passing from eq. (8) to eq. (13), we have dropped the p equations with the subscript $s = q$, because, as is demonstrated above, they can be considered to be redundant.

Finally, a complete linear system can be written as :

(1) $x^m_{\cdot i} = \sum_n a^i_{mn} x^n_{i\cdot} + y^m_i$,

where $m = 1, 2, \ldots, p : i = 1, 2, \ldots, q$.

(9) $\sum_i x^m_{i\cdot} = \sum_i x^m_{\cdot i}$ $(\equiv x^m_{\cdot\cdot})$,

where $m = 1, 2, \ldots, p$.

(13) $\sum_r (\triangle x^m_{\cdot r} M^m_{ir}) - \sum_r (\triangle x^m_{r\cdot} N^m_{ri}) = 0$,

where $m = 1, 2, \ldots, p : i = 1, 2, \ldots, (q\text{-}1)$.

Figure 5

Relations between Equations and Unknowns

a) Equations

(1) … pq
(9) … p
(13) … $p\ (q-1)$
} $2pq$

b) Unknowns

$x^m_{i\cdot}$ … pq
$x^m_{\cdot j}$ … pq
} $2pq$

The corresponding changes in all intraregional flows $\triangle x^m_{ii}$, and interregional flows $\triangle x^m_{ij}$, can be determind by inserting the previously computed values of $\triangle x^m_{i\cdot}$ and $\triangle x^m_{\cdot j}$ into equations (4) and (5), or eq. (6)′.

IV

1. On the Concept of Entropy[7)]

The measure of the uncertainty was given by Shannon as

(14) $S\ (p_1, p_2, \ldots, p_n) = -k \sum_{i=1}^{n} p_i\, l_n\, p_i$,

where $k \geq 0$.

This is defined to be the *entropy* of the probability distribution $p_1, p_2, \ldots, p_n$.

7) In this section we draw heavily upon Wilson (9), Appendix 1. However, since his explanation has some troubles in using notations, we have somewhat modified it.

The proof that this is a unique, unambiguous measure of uncertainty can be sketched as follows :

We want a quantity $S\ (p_1, p_2, \ldots, p_n)$ to represent the uncertainty associated with a probability distribution $p_1, p_2, \ldots, p_n$. Only three conditions have to be satisfied :

1) S is a continuous function of the p_i.
2) If all p_i are equal,

$$(15)\quad A(n) = S(\frac{1}{n}, \frac{1}{n}, \ldots\ldots, \frac{1}{n})$$

is an increasing function of n.[8]

3) Suppose events are grouped in various ways, and let

$$(16)\quad \left.\begin{array}{l} w_1 = p_1 + p_2 + \ldots + p_k \\ w_2 = p_{k+1} + \ldots\ldots\ldots + p_l \\ \vdots \qquad \vdots \qquad\qquad\qquad \vdots \\ w_h = p_{m+1} + \ldots\ldots\ldots + p_n. \end{array}\right\}$$

Then $p_1 \mid w_1, p_2 \mid w_1, \ldots\ldots$ are the conditional probabilities of the events $(x_1, x_2, \ldots, x_k), (x_{k+1}, \ldots\ldots, x_l), \ldots\ldots, (x_{m+1}, \ldots, x_n)$.[9]

We require that the following *composition law* be satisfied :

$$(17)\quad \begin{aligned} S\ (p_1, p_2, \ldots, p_n) = {} & S\ (w_1, w_2, \ldots, w_h) \\ & + w_1\ S\ (p_1 \mid w_1, p_2 \mid w_1, \ldots, p_k \mid w_1) \\ & + w_2\ S\ (p_{k+1} \mid w_2, \ldots, p_l \mid w_2) + \ldots\ldots\ldots\ldots \\ & + w_h\ S\ (p_{m+1} \mid w_h, \ldots, p_n \mid w_h). \end{aligned}$$

Because of condition 1), we only need determine S for rational values of w_j,

$$(18)\quad w_j = \frac{n_j}{\sum_{j=1}^{h} n_j},$$

where the n_j are integers.

The following chart may be helpful to understand this situation.

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8) Each event corresponds to the sample point which has the equal probability.

9) Condition 3) is introduced to give the property of the probability to the function S.

Figure 6
Relations between Events and Composite Events

a) Name of Events

$(x_1, x_2, \ldots, x_k), (x_{k+1}, \ldots, x_l), \ldots, (x_{j1}, \ldots, x_{jnj}), \ldots, (x_{m+1}, \ldots, x_n)$.

b) Number of Events in Group j $(j = 1, 2, \ldots, h)$

$n_1, \quad n_2, \ldots, \quad n_j, \ldots, \quad n_h$.

c) Name of Composite Events

$X_1, \quad X_2, \ldots, \quad X_j, \ldots, \quad X_h$.

d) Probability of Composite Events

$w_1, \quad w_2, \ldots, \quad w_j, \ldots, \quad w_h$.

Then, we can view this as follows: x_j can occur n_j times out of $\sum_{j=1}^{h} n_j$ equal possibilities. That is, we can consider our *events* $x_1, x_2, \ldots, x_h$ as themselves composite events out of $n_1, n_2, \ldots, n_h$ equal alternatives. Thus, condition 3) gives

(19) $$S(w_1, w_2, \ldots, w_h) + \sum_j w_j S(p_{j1} \mid w_j, \ldots, p_{jnj} \mid w_j) = S(p_1, p_2, \ldots, p_n).$$

In particular, we can choose all n_j equal to m,[10] so eq. (19) reduces to

(20) $$A(h) + A(m) = A(mh).$$

It can then be shown that the only function which satisfies this and condition 2) is

(21) $$A(m) = kl_n(m),$$

where $k \geq 0$.

Substitute from eq. (21) into eq. (19), to obtain

(22) $$\begin{aligned} S(w_1, w_2, \ldots, w_h) &= S(p_1, p_2, \ldots, p_n) - \sum_j w_j S(p_{j1} \mid w_j, \ldots, p_{jn} \mid w_j) \\ &= k\, l_n\, n - k \sum_j w_j\, l_n\, n_j \\ &= -k\left(\sum_j w_j\, l_n\, n_j - l_n\, n\right) \end{aligned}$$

10) This means that in eq. (18), $w_j = \dfrac{n_j}{\sum_{j=1}^{n} n_j} = \dfrac{m}{hm} = \dfrac{1}{h}$.

$$= -k(\sum_j w_j \, l_n \, n_j - \sum_j w_j \, l_n \, n) \text{ since } \sum_j w_j = 1,$$

$$= -k \sum_j w_j \, l_n \, (n_j - n)$$

$$= -k \sum_j w_j \, l_n \frac{n_j}{n}$$

$$= -k \sum_j w_j \, l_n \, w_j \qquad \text{using eq. (18).}$$

Thus, in general, the entropy of a probability distribution can be defined as

(14) $S(p_1, p_2, \ldots, p_n) = -k \sum_{j=1}^{n} p_j \, l_n \, p_j.$

This is a unique, unambiguous criterion for the *amount of uncertainty* represented by a discrete probability distribution.[11]

It also agrees with our intuitive notions that a broad distribution represents more uncertainty than does a sharply peaked one. Let X be a random variable which can take values X_1, X_2, and X_3 with probabilities p_1, p_2, and p_3, respectively.

Figure 7
Hypothetical Example

a)	Name of Events :	X_1 ,	X_2 ,	X_3.
b)	Probabilities :	p_1 ,	p_2	p_3.

If we are confident that X_1 will surely occur, then it is quite natural that we would assign to p_1 and 0 to the other probabilities, p_2, and p_3. In this case, the entropy as a measure of the uncertainty can be evaluated as

(23) $S(1, 0, 0) = -k(1 \, l_n \, 1 + 0 \, \underset{p_2 \to 0}{l_n \, p_2} + 0 \, \underset{p_3 \to 0}{l_n \, p_3}) = 0.$

On the other hand, if we are not quite sure that any X_i will most probably occur, then we would assign the equal probability ($\frac{1}{3}$ in this case) to every random variable ($X_i : i = 1, 2, 3$).

11) As for a continuous probability distribution, the entropy can be defined by
$S(X) = -\int_{-\infty}^{\infty} f(X) \, l_n \, (X) dX.$

The entropy can be evaluated as

(24) $$S(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = -k(\frac{1}{3}\, l_n \frac{1}{3} + \frac{1}{3}\, l_n \frac{1}{3} + \frac{1}{3}\, l_n \frac{1}{3}) = 1.$$ [12]

It is now evident how the entropy plays a role of criterion for the amount of uncertainty, which agrees with our intuitive notions and meets the properties included in the concept of the probability.

2. Derivation of the Gravity Model using Entropy-maximizing Methods

2.1. The Gravity Model

Any derivation of the gravity model is based on analogies between spatial interaction in geography and spatial interaction in classical physics. Let X_i^m and Y_j^m be *masses* of commodity m related to the origin and destination of a spatial interaction between regions i and j. The transport cost of a unit of commodity m is defined to be c_{ij}^m and this can be considered to be a *distance*.

A strictly Newtonian interaction would then be an x_{ij}^m defined by

(25) $$x_{ij}^m = K_m \frac{X_i^m \ Y_j^m}{(c_{ij}^m)^2},$$

where K^m is a normalizing factor which ensures that

(26) $$\sum_i \sum_j x_{ij}^m = \sum_i X_i^m = \sum_j Y_j^m = X^m.$$

That is,

(27) $$K^m = \frac{X^u}{\sum_i \sum_j [X_i^m \ Y_j^m \ / \ (c_{ij}^m)^2].}$$

The first development of this model is to argue that geographic spatial interaction for commodity flows may well be governed by a general distance function other than the inversesquare law.

The modified gravity model[13] is then

12) The parameter "k" is so defined to make S unity.

13) Note that we have allowed in our notation for the possibility of a different function for each commodity group.

(28) $x_{ij}^{m} = K^{m} X_{i}^{m} Y_{j}^{m} f^{m} (c_{ij}^{m}),$

where $f^{m} (c_{ij}^{m})$ is some decreasing function of c_{ij}^{m},

K^{m} is now calculated from

(29) $K^{m} = \dfrac{X^{m}}{\sum_{i} \sum_{j} X_{i}^{m} Y_{j}^{m} f^{m} (c_{ij}^{m})}.$

2.2. Classification of the Basic Cases

Further development is possible, but, as a preliminary, we must interpret our terms very carefully. Strictly, a model of interregional commodity flows provides estimates of x_{ij}^{m}, and hence, of $x_{i.}^{m}$, $x_{.j}^{m}$, and $x_{..}^{m}$. However, $x_{..}^{m}$, and possibly $x_{i.}^{m}$ and $x_{.j}^{m}$, may be estimated directly from independent models, and in our notation we have called such estimates X^{m}, X_{i}^{m}, and Y_{j}^{m}, respectively.

There are four possible cases to be studied :

Case (1) there is an independent estimate of X^{m}, but not of X_{i}^{m} or Y_{j}^{m}.

Case (2) there is an independent estimate of X_{i}^{m} (which determines X^{m}), but not of Y_{j}^{m}.

Case (3) there is an independent estimate of Y_{j}^{m} (which determines X^{m}), but not of X_{i}^{m}.

Case (4) thers are independent estimates of both X_{i}^{m} and Y_{j}^{m} (made in such a way that they determine X^{m} and that $\sum_{i} X_{i}^{m} = X^{m}$ and $\sum_{j} Y_{j}^{m} = X^{m}$).

We can now carry out a further appraisal of the Newtonian form of the gravity model presented in eq. (28). Note that in eqs. (28) and (29), X_{i}^{m} should be replaced by $x_{i.}^{m}$, and Y_{j}^{m} by $x_{.j}^{m}$, in cases where they are not independently estimated. Since an estimate of X^{m} is assumed to exist in all cases, an equation of the form of eq. (29) can always be used to estimate K^{m}.

Thus eqs. (28) and (29) represent the Newtonian gravity model for case (4) and can easily be solved directly for x_{ij}^{m}. For each of cases (1), (2), (3), the modified versions of eqs. (28) and (29) lead to quadratic equations in x_{ij}^m which cannot easily be solved.

Consider case (4), which may be called the origin-destination-constrained model, because the following equations should be satisfied :

(30) $\sum_j x_{ij}^m = X_i^m$,

(31) $\sum_i x_{ij}^m = Y_j^m$.

Then, we can find a set of normalizing factors to replace the single factor K^m which will ensure that eqs. (30) and (31) are always satisfied.

Define a set of factors A_i^m and B_j^m and then modify eq. (28) to read

(32) $x_{ij}^m = A_i^m B_j^m X_i^m Y_i^m f^m (c_{ij}^m)$.

The factors A_i^m and B_j^m can be calculated by substituting x_{ij}^m from eq. (32) into eqs. (30) and (31), respectively.

This gives[14)]

(33) $A_i^m = [\sum_j B_j^m Y_j^m f^m (c_{ij}^m)]^{-1}$,

(34) $B_j^m = [\sum_i A_i^m X_i^m f^m (c_{ij}^m)]^{-1}$,

and eqs. (33) and (34) can be solved iteratively.

2.3. Entropy associated with the Commodity Flows

The entropy maximizing principle offers a general tool. If a set of variables are to be estimated, such as the flows x_{ij}^m, and if the known constraints on x_{ij}^m can be expressed in equation from, then the entropy of a probability distribution associated with x_{ij}^m can be maximized and a maximum probability estimate of x_{ij}^m obtained. Before we use this general tool to integrate the gravity and input-output models, it will be more useful to show how to gravity model presented in Section 4. 2. 1. can be derived, and this will further

14) A_i^m signifies the constant associated with origin, while B_j^m signifies the constant associated with destination. Note also that c_{ij}^m should be interpreted as a general measure of impedance, as travel time, as cost, or more effectively as some weighted combination of such factors sometimes referred to as a *generalized cost*.

deepen our understanding of the gravity model itself.

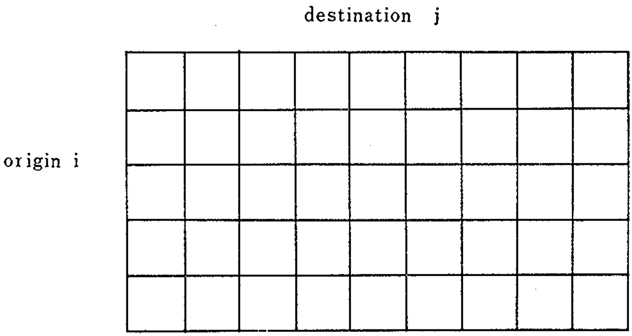
In addition to the constraints (30) and (31), we assume that a total amount C^m is spent on transporting commodity m.
That is, as a cost constraint,

$$(35) \quad \sum_i \sum_j x^m_{ij}\, c^m_{ij} = C^m.$$

Let us find the matrix $\{x^m_{ij}\}$ which has the greatest number of states, say $W(\{x^m_{ij}\})$, subject to the constraints (30), (31), and (35). The number of states which give rise to a matrix $\{x^m_{ij}\}$ can be obtained as follows:

Suppose X^m is the total amount of commodity m, (*i. e.*, $X^m = \sum_i \sum_j x^m_{ij}$). How many assignments of commodity m to *boxes* of Figure 8 give rise to $\{x^m_{ij}\}$?

Figure 8
Origin-Destination Table
(in a singlecommodity case)



Firstly we can select x^m_{11} from X^m, x^m_{12} from $X^m - x^m_{11}$, etc., and so the number of possible assignments, or states, is the number of ways of selecting x^m_{11} from X^m ($X^m\, C\, x^m_{11}$), multiplied by the number of ways of selecting x^m_{12} from $X^m - x^m_{11}$ ($X^m - x^m_{11}\, C\, x^m_{12}$), etc. Thus,[15]

15) This result is independent of the order in which the boxes of Figure 8 are considered.

(36) $$W(\{x_{ij}^m\}) = \frac{X^m!}{x_{11}^m!(X^m - x_{11}^m)!} \cdot \frac{(X^m - x_{11}^m)!}{x_{12}^m!(X^m - x_{11}^m - x_{12}^m)!} \cdots\cdots\cdots$$

$$= \frac{X^m!}{\prod_{i,j} x_{ij}^m!}$$

We now maximize $W(\{x_{ij}^m\})$ subject to eqs. (30), (31), and (35) in order to find the most probable $\{x_{ij}^m\}$. In fact, any monotonic function of $W(\{x_{ij}^m\})$ can be used to give the same result, and for convenience we maximize $l_n(W(\{x_{ij}^m\}))$ --- we write this as $l_n(\{W_{ij}^m\})$ hereafter --- subject to eqs. (30), (31), and (35).

We now have to show that the measure of uncertainty, which is restated here for convenience,

(14) $$S(p_1, p_2, \ldots, p_n) = -k \sum_i p_i \, l_n \, p_i$$

is the same as that introduced above.

Define

(37) $$p^{m}_{ij} = \frac{x_{ij}^m}{X^m}.$$

Then, from eq. (36)

(38) $$l_n(\{W_{ij}^m\}) = l_n X^m! - \sum_i \sum_j l_n \, x_{ij}^m!$$

$$= l_n X^m! - \sum_i \sum_j (x_{ij}^m \, l_n \, x_{ij}^m - x_{ji}^m)$$

after the use of Stirling's approximation for $l_n \, x_{ij}^m!$[16)]

This can be written in terms of p_{ij} as

(38)′ $$l_n(\{W_{ij}^m\}) = l_n X^m! - \sum_i \sum_j [p_{ij}^m X^m (l_n \, p_{ij}^m + l_n X^m) - p_{ij}^m X^m]$$

$$= l_n X^m! - X^m \sum_i \sum_j p_{ij}^m \, l_n \, p_{ij}^m - (X^m \, l_n X^m - X^m) \sum_i \sum_j p_{ij}^m$$

16) Stirling's approximation can be used to estimate the factorial terms, *i. e.*, $l_n N! = N \, l_n N - N$.

$$= (l_n X^m! - X^m l_n X^m + X^m) - X^m \sum_i \sum_j p_{ij}^m l_n p_{ij}^m.$$

Hence, maximizing

$$S(p_{11}^m, p_{12}^m, \ldots\ldots) = -\sum_i \sum_j p_{ij}^m l_n p_{ij}^m$$

subject to eqs. (30), (31), and (35) --- which can be expressed as constraints on the p_{ij}^m using our definition (37) --- will lead to an estimate of x_{ij}^m which is the same as that given above. In short, $l_n(\{W_{ij}^m\})$ and S are linearly related.

2.4. Formulation and its Solution

Our problem under study can be formulated as follows : Maximize[17)]

(39) $S = -\sum_i \sum_j l_n x_{ij}^m!$

subject to

(30) $\sum_j x_{ij}^m = X_i^m$

(31) $\sum_i x_{ij}^m = Y_j^m$

(35) $\sum_i \sum_j x_{ij}^m \cdot c_{ij}^m = C^m.$

First-order Conditions

To obtain the set of x_{ij}^m which maximizes eq. (39) subject to the constraints (30), (31), and (35), the Lagrangean form has to be maximized :

(40) $$L = -\sum_i \sum_j l_n x_{ij}^m! + \sum_i \lambda_i^{(1)}(X_i^m - \sum_i x_{ij}^m) + \sum_i \lambda_j^{(2)}(Y_i^m - \sum_i x_{ij}^m)$$
$$+ \mu^m (C^m - \sum_i \sum_j x_{ij}^m \cdot c_{ij}^m)$$

where $\lambda_i^{(1)}$, $\lambda_j^{(2)}$, and μ^m are Lagrangean multipliers.

The first-order conditions are[18)] :

17) It is allowed to use this form of S, since maximizing $l_n(\{W_{ij}^m\})$ -- the left hand side of eq. (38) -- gives the same answer as using $-\sum_i \sum_j l_n x_{ij}^m!$ -- the second term of the right hand side of eq. (38), so long as $\sum_i \sum_j x_{ij}^m = X^m =$ constant.

18) For the derivation of eq. (41), the Stirling's approximation is also used as follows : $l_n N! = N l_n N - N, \partial l_n N! / \partial N = l_n N + N \cdot \frac{1}{N} - 1 = l_n N.$

$$(41)\quad \frac{\partial L}{\partial x^m_{ij}} = -\, l_n\, x^m_{ij} - \lambda^{(1)}_i - \lambda^{(2)}_j - u^m\, c^m_{ij} = 0,$$

$$(42)\quad \frac{\partial L}{\partial \lambda^{(1)}_i} = X^m_i - \sum_j x^m_{ij} = 0,$$

$$(43)\quad \frac{\partial L}{\partial \lambda^{(2)}_j} = Y^m_j - \sum_i x^m_{ij} = 0,$$

$$(44)\quad \frac{\partial L}{\partial \mu^m} = C^m - \sum_i \sum_j x^m_{ij}\, c^m_{ij} = 0.$$

From eq. (41)

$$(45)\quad x^m_{ij} = \exp\,(-\,\lambda^{(1)}_i - \lambda^{(2)}_j - u^m\, c^m_{ij}\,).$$

Substitute in eqs. (42) and (43) to obtain $\lambda^{(1)}_i$ and $\lambda^{(2)}_j$:

$$(46)\quad X^m_i = \sum_j x^m_{ij} = \sum_j \exp\,(-\,\lambda^{(1)}_i - \lambda^{(2)}_j - \mu^m\, c^m_{ij})$$

$$= \exp\,(-\,\lambda^{(1)}_i)\,[\sum_j \exp\,(-\,\lambda^{(2)}_j - \mu^m\, c^m_{ij})].$$

Hence,

$$(47)\quad \exp\,(-\,\lambda^{(1)}_i) = X^m_i\,[\sum_j \exp\,(-\,\lambda^{(2)}_j - \mu^m\, c^m_{ij})]^{-1}.$$

Similarly,

$$(48)\quad Y^m_j = \sum_i x^m_{ij} = \sum_i \exp\,(-\,\lambda^{(1)}_i - \lambda^{(2)}_j - \mu^m\, c^m_{ij})$$

$$= \exp\,(-\,\lambda^{(2)}_j)\,[\sum_i \exp\,(-\,\lambda^{(1)}_i - \mu^m\, c^m_{ij})].$$

$$(49)\quad \exp\,(-\,\lambda^{(2)}_j) = Y^m_j\,[\sum_i \exp\,(-\,\lambda^{(1)}_i - \mu^m\, c^m_{ij})]^{-1}.$$

To obtain the final result in more familiar form, write

$$(50)\quad A^m_i = \frac{\exp\,(-\,\lambda^{(1)}_i)}{X^m_i},$$

$$(51)\quad B^m_j = \frac{\exp\,(-\,\lambda^{(2)}_j)}{Y^m_j},$$

and then

(52) $x_{ij}^m = A_i^m B_j^m X_i^m Y_j^m \exp(-\mu^m c_{ij}^m)$,

where, using eqs. (47), (49), (50), and (51)

(53) $A_i^m = [\sum_j B_j^m Y_j^m \exp(-\mu^m c_{ij}^m)]^{-1}$,

(54) $B_j^m = [\sum_i A_i^m X_i^m \exp(-\mu^m c_{ij}^m)]^{-1}$.

This model is now equivalent to that given in eqs. (32) through (34), with the negative exponential function $\exp(-\mu^m c_{ij}^m)$ replacing the general function $f^m(c_{ij}^m)$.

The statistical derivation constitutes a new theoretical base for the gravity model. This statistical theory is effectively saying that, given total amounts of origins and destinations for each zone for a homogeneous commodity m (*i. e.*, X_i^m, Y_j^m), given the costs of transporting between each zone (*i. e.*, c_{ij}^m), and given that there is some fixed total expenditure on transport (*i. e.*, C^m), then there is a most probable distribution of commodity flows between zones, and this distribution is the same as the one normally described as *gravity model* distribution.

2.5. Sufficiency-test for the Solution

The sufficient conditions for distinguishing maxima from minima require negative definiteness of the bordered Hessian-like matrix. For the purpose of the sufficiency test, let the number of regions be q as in Section 3. Then, our objective function given by eq. (39) can be written in more general form as

(39)′ $-\sum_i \sum_j l_n\, x_{ij}^m! \longrightarrow f(x_{11}^m, x_{12}^m, \ldots\ldots\ldots\ldots, x_{qq}^m)$

Our constraints given by eqs. (30), (31), and (35) can also be rewritten as

(30)′
$$\begin{cases} X_l^m - \sum_j x_{lj}^m = 0 \longrightarrow h^1(x_{11}^m, x_{12}^m, \ldots\ldots, x_{lq}^m) = 0, \\ \vdots \\ X_q^m - \sum_j x_{qj}^m = 0 \longrightarrow h^q(x_{q1}^m, x_{q2}^m, \ldots\ldots, x_{qq}^m) = 0, \end{cases}$$

$$(31)' \begin{cases} Y_1^m - \sum_i x_{i1}^m = 0 \longrightarrow h^{q+1}(x_{11}^m, x_{21}^m, \ldots, x_{q1}^m) = 0, \\ \vdots \\ Y_q^m - \sum_i x_{iq}^m = 0 \longrightarrow h^{2q}(x_{1q}^m, x_{2q}^m, \ldots, x_{qq}^m) = 0, \end{cases}$$

$$(35)' \quad C^m - \sum_i \sum_j x_{ij}^m c_{ij}^m = 0 \longrightarrow h^{2q+1}(x_{11}^m, x_{12}^m, \ldots\ldots, x_{qq}^m) = 0.$$

Since we have assumed that $\sum_i X_i^m = \sum_j Y_j^m$,[19] the equation system given by eqs. (30)′ and (31)′ is no longer independent. Thus, we can drop any one of them--say, the last equation in (31)′ --without loss of generality.

Now a system of $2q$ linear homogeneous equations (*i. e.*, q of (30)′ plus (q-1) out of (31)′ plus 1 of (35)′) in q^2 variables (*i. e.*, $x_{11}^m, x_{12}^m, \ldots, x_{1q}^m, \ldots\ldots\ldots, x_{qq}^m$) can be represented as follows[20]:

$$(55) \quad \begin{Bmatrix} h^1(x_{11}^m, x_{12}^m, \ldots\ldots, x_{1q}^m) = 0 \\ \vdots \\ h^{2q-1}(x_{1q-1}^m, \ldots, x_{q,q-1}^m) = 0 \\ h^{2q}(x_{11}^m, x_{12}^m, \ldots\ldots, x_{qq}^m) = 0 \end{Bmatrix} \text{ or } h^i(X) = 0 \text{ where } i = 1, 2, ., 2q.$$

The Jacobian matrix of this system of constraints, $h^i(X) = 0$, is defined as[21]

$$(56) \quad J = \begin{pmatrix} \partial h^1/\partial x_{11}^m, \ldots\ldots, \partial h^1/\partial x_{qq}^m \\ \partial h^2/\partial x_{11}^m, \ldots\ldots, \partial h^2/\partial x_{qq}^m \\ \vdots \\ \partial h^{2q}/\partial x_{11}^m, \ldots, \partial h^{2q}/\partial x_{qq}^m \end{pmatrix} = \begin{pmatrix} h^1_{x^m_{11}}, \ldots\ldots, h^1_{x^m_{qq}} \\ h^2_{x^m_{11}}, \ldots\ldots, h^2_{x^m_{qq}} \\ \\ h^{2q}_{x^m_{11}}, \ldots\ldots, h^{2q}_{x^m_{qq}} \end{pmatrix}$$

Let V be the $q^2 \times q^2$ matrix of elements

19) Otherwise, the system of equations given by eqs. (30) and (31) becomes inconsistent.

20) The number of equations (2q) should be less than the number of variables (q^2). The existence of more equations than variables always implies either redundant equations (which can be dropped) or an inconsistent system in which not all equations can be met simultaneously. For this reason, $q \geq 2$.

21) The rank of the Jacobian matrix must be $2q$ in this case.

$$(57)\quad v_{ij,\,kl} = \frac{\partial^2 f}{\partial x_{ij}^m\, \partial x_{kl}^m} - \sum_{i=1}^{q} \lambda_i^{(1)}\, h^i_{x^m_{ij},\, x^m_{kl}} - \sum_{j=q+1}^{2q-1} \lambda_j^{(2)}\, h^j_{x^m_{ij},\, x^m_{kl}} - \mu^m\, h^{2q}_{x^m_{ij},\, x^m_{kl}},$$

which means the second partials or cross-partials of f (or S) less the sum of those same partials or cross-partials in each of the constraints, each multiplied by the Lagramgean multiplier for that constraint.

Thus, the bordered Hessian-like matrix is

$$(58)\quad \overline{V}^* = \left(\begin{array}{c:c} \underset{(2q\times 2q)}{O} & \underset{(2q\times q^2)}{J^*} \\ \hdashline \underset{(q^2\times 2q)}{J^{*\prime}} & \underset{(q^2\times q^2)}{V^*} \end{array}\right)$$

where asterisks indicate evaluation at a point satisfying the first-order conditions for a maximum or a minimum. Then, for $d^2 f^* < 0$ (and hence a maximum at X^*), the last $q^2 - 2q$ principal minors of $\overline{V}^*$ must alternate in sign, with the first having the sign $(-1)^{2q+1}$.

Obviously, for problems in which the number of constraints and the number of variables are large, the work involved in evaluating principal minors becomes immense, both because each determinant is large and because there may be many of them.

We conclude with a specific example. Let us examine the three regions' case (*i. e.*, $q = 3$). The bordered Hessian-like matrix in question becomes[22] as follows (eq. (59)).

Let us consider this situation from a somewhat different viewpoint. Namely consider the property of our objective function given by eq. (39) as $S = -\sum_i \sum_j l_n\, x_{ij}^m\,!$. Let it be $f(X)$ in general. It is quite clear that maxima should be separated from minima by the slope of the function f. Investigation of the signs of the principal minors of the Hessian matrix of second partial derivatives evaluated at X^* (when $df = 0$) is precisely an investigation of the function for convexity or concavity in the neighborhood of X^*. So, if a point

22) As usual, asterisks denote that elements of the matrix are evaluated at the point that satisfy the first-order conditions, given by eqs. (41) through (44).

$$
(59)\ \bar{V}^{*}=\left[\begin{array}{cccccc|ccccccccc}
0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & & & \vdots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
\vdots & \vdots & & \ddots & & \vdots & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
\vdots & \vdots & & & \ddots & \vdots & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & -C^{m}_{11} & -C^{m}_{12} & -C^{m}_{13} & -C^{m}_{21} & -C^{m}_{22} & -C^{m}_{23} & -C^{m}_{31} & -C^{m}_{32} & -C^{m}_{33} \\
\hline
-1 & 0 & 0 & -1 & 0 & -C^{m}_{11} & -\frac{1}{x^{m*}_{11}} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & 0 & 0 & 0 & -1 & -C^{m}_{12} & 0 & -\frac{1}{x^{m*}_{12}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & 0 & 0 & 0 & 0 & -C^{m}_{12} & 0 & 0 & -\frac{1}{x^{m*}_{13}} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & -1 & 0 & -1 & 0 & -C^{m}_{21} & \vdots & \vdots & \vdots & \ddots & & & & & \vdots \\
0 & -1 & 0 & 0 & -1 & -C^{m}_{22} & \vdots & \vdots & \vdots & & \ddots & & & & \vdots \\
0 & -1 & 0 & 0 & 0 & -C^{m}_{23} & \vdots & \vdots & \vdots & & & \ddots & & & \vdots \\
0 & 0 & -1 & -1 & 0 & -C^{m}_{31} & \vdots & \vdots & \vdots & & & & \ddots & & \vdots \\
0 & 0 & -1 & 0 & -1 & -C^{m}_{32} & \vdots & \vdots & \vdots & & & & & \ddots & \vdots \\
0 & 0 & -1 & 0 & 0 & -C^{m}_{33} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & -\frac{1}{x^{m*}_{33}}
\end{array}\right]
$$

X^* is found for which $df^* = 0$, and if it is known that f is a concave function in the neighborhood of X^*, then we know that X^* represents a relative maximum. This relation-ship can be shown through the use of Taylor's series.

Let $X^* = (x^{m*}_{11}, x^{m*}_{12}, \ldots\ldots\ldots\ldots)'$ be a point at which the first total differential of the function is zero --- that is, $df^* = 0$; and let $(X^* + dX)$ represent the *nearby* point $(x^{m*}_{11} + dx^{m}_{11}, x^{m*}_{12} + dx^{m}_{12}, \ldots\ldots\ldots)'$. Then, the Taylor expansion is

$$(60)\quad f(X^* + dX) = f(X^*) + df^* + \left(\frac{1}{2!}\right) d^2 f^* + \left(\frac{1}{3!}\right) d^3 f^* + \cdots.$$

Thus a second-order approximation to the change in the function as a result of the displacement dX is given by

$$(61)\quad f(X^* + dX) - f(X^*) \cong df^* + \frac{1}{2}\, d^2 f^*.$$

Since X^* has been chosen such that $df^* = 0$, the sign of the change in the value of $f(x^{m}_{11}, x^{m}_{12}, \ldots\ldots\ldots)$ depends entirely on the sign of $d^2 f^*$, the second total differential evaluated at X^*. At, or very near the stationary point,

$$(62)\quad f(X^* + dX) - f(X^*) = df^* + \frac{1}{2}\, d^2 f^*$$
$$= 0 + \frac{1}{2} \sum_{ik}\sum_{jl} \frac{\partial^2 f}{\partial (x_{ij}^m)\ \partial (x_{kl}^m)} (dx_{ij}^m)\ (dx_{kl}^m).$$

But, we have seen that

$$(39)'\quad f(X) = -\sum_i \sum_j l_n\, x_{ij}^m\,!,$$

and

$$(63)\quad \frac{\partial f(X)}{\partial\, x_{ij}^m} = -l_n\, x_{ij}^m \text{ [23]},$$

so

$$(64)\quad \frac{\partial^2 f(X)}{\partial\, x_{ij}^m\ \partial\, x_{kl}^m} = \begin{cases} -\dfrac{1}{x_{ij}^m} & (\text{if } i = k \text{ and } j = 1), \\ 0 & (\text{if } i \neq k \text{ or } k \neq 1). \end{cases}$$

Substituting in eq. (62), we have

$$(65)\quad f(X^* + dX) - f(X^*) = -\frac{1}{2} \sum_i \sum_j \frac{(dx_{ij}^m)^2}{x_{ij}^m}$$
$$= -\frac{1}{2} \sum_i \sum_j \left(\frac{dx_{ij}^m}{x_{ij}^m}\right)^2 x_{ij}^m,$$

where $\dfrac{dx_{ij}^m}{x_{ij}^m}$ is the relative change in x_{ij}^m away from the most probable distrbution. From this result, we can state that so long as all x_{ij}^m are positive, $f(X + dX) - f(X^*) < 0$, and hence X^* represents a local maximum --- the function is concave in the neighborhood of X^*.

However, if some Lagrangean multipliers $\lambda_i^{(1)}$, $\lambda_j^{(2)}$, u^m, take on positive values, the stationary point X^* derived from the unconstrained maximization

23) See eq. (41) and footnote 18. As for the three regions' case, see eq. (59).

problem does not meet all the constraints given by eqs. (30), (31), and (35). In this sense, Lagrangean multipliers ($\lambda_i^{(1)}$, $\lambda_j^{(2)}$, μ^m) provide useful information about the constraints h^i ($\overset{m}{x}_{11}$, $\overset{m}{x}_{12}$, ...) = 0.

When we consider the constraints explicitly, using the Lagrangean form, their values evaluated at a stationary point (X^*) --- now the dimension of X^* vector is augmented by the number of constraints --- give the partial derivatives of our objective function $f(X)$ evaluated at X^*, with respect to the constraint constants. Loosely speaking, they give approximations to the amounts that the optimum value of $f(X)$ will change for a unit change in the constraint constants. Therefore, by examining the values of Lagrangean multipliers, we can easily check how effectively each constraint works in the maximization problem.

3. Integration of the Gravity and Input-Output Models

In the gravity-model approach to case (1), we assumed that an independent estimate of X^m did exist, though there was no such estimate for X_i^m and Y_j^m. For this particular development of the integrated model, we assume that there is no such estimate of X^m, and this brings us into line with the assumptions of Leontief and Strout, which we referred to in Section 3. Thus, the case (1) model to be developed here represents the modification to the Leontief-Strout model brought about by integrating the gravity and input-output models using *entropy-maximizing principles*, but otherwise no new assumptions are made.[24]

The only constraints, then, are eq. (35), which is restated here for convenience :

---

24) Case (4) model --- which we called the origin-destination-constrained model in Section 4. 2. 2. and considered in Section 4. 2. 4. --- is the only one which offers a simple estimate of x_{ij}^m. The other cases, (1), (2), and (3) --- the modified versions of eqs. (28) and (29) --- lead to quadratic equations in x_{ij}^m, although some iterative-solution-procedure could be devised. In this Section 4. 3., we only refer to case (1) as representative of the rest.

(35) $\sum_i \sum_j x_{ij}^m \, c_{ij}^m = C^m,$

and the Leontief-Strout eq. (1) as a constraint on x_{ij}^m, and so we rewrite it in terms of the x_{ij}^m as

(1)′ $\sum_j x_{ij}^m = \sum_n a_{mn}^i \sum_i x_{ij}^n + y_i^m.$

We now have to maximize the entropy of the probability distribution associated with x_{ij}^m (with m varying now as well as i & j). The problem can be formulated as follows : [25)]

Maximize our entropy S defined by

(66) $S(X) = -\sum_i \sum_j \sum_m x_{ij}^m \, l_n \, x_{ij}^m,$

subject to eqs. (35) and (1)′.

To solve the problem, we form the Lagrangean form L :

(67) $L = -\sum_i \sum_j \sum_m x_{ij}^m \, l_n \, x_{ij}^m + \sum_i \sum_m \gamma_i^m \, (y_i^m + \sum_n a_{mn}^i \sum_j x_{ij}^n - \sum_j x_{ji}^m)$

$$+ \sum_m \mu^m \, (C^m - \sum_i \sum_j x_{ij}^m \cdot c_{ij}^m)$$

where γ_i^m is the set of Lagrangean multipliers associated with eq. (1)′ and μ^m the set associated with eq. (35). We now obtain an estimate of x_{ij}^m by solving the first-order conditions :

(68) $\dfrac{\partial L}{\partial x_{ij}^m} = -\, l_n \, x_{ij}^m - 1 + \sum_m \gamma_i^m \, a_{mn}^i - \gamma_j^m - \mu^m \, c_{ij}^m = 0,$

(69) $\dfrac{\partial L}{\partial \gamma_i^m} = y_i^m + \sum_n a_{mn}^i \sum_j x_{ij}^n - \sum_j x_{ij}^m = 0,$

(70) $\dfrac{\partial L}{\partial \mu^m} = C^m - \sum_i \sum_j x_{ij}^m \cdot c_{ij}^m = 0.$

Eq. (68) gives

(71) $x_{ij}^m = \exp\,(\sum_m \gamma_i^m \, a_{mn}^i - \gamma_j^m - \mu^m \, c_{ij}^m),$

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25) Eq. (66) can be derived from eq. (38). It is convenient to use this form of S, since $-\sum_i \sum_j l_n \, x_{ij}^m\,!$ appears to cause conceptual difficulties if x_{ij}^m is noninteger.

where a 1 has been absorbed into the multiplier γ_j^m, without loss of generality.

Now, μ^m is obtained by substituting x_{ij}^m from eq. (71) into eq. (70), and similarly γ_j^m is found by substituting x_{ij}^m from eq. (71) into eq. (69). This gives

$$(72)\quad \sum_i \exp\left(\sum_m \gamma_i^m a_{mn}^i - \gamma_j^m - \mu^m c_{ij}^m\right) - \sum_n a_{mn}^i \sum_j \exp\left(\sum_m \gamma_i^n a_{mn}^i - \gamma_j^m - \mu^m c_{ij}^m\right) - y_i^m = 0.$$

Then,

$$(72)'\quad \exp(-\gamma_j^m) \sum_i \exp\left(\sum_m \gamma_i^m a_{mn}^i - \mu^m c_{ij}^m\right) - \sum_n a_{mn}^i \exp\left(\sum_m \gamma_i^m a_{mn}^i\right) \sum_j \exp(-\gamma_j^m - \mu^m c_{ij}^m) - y_i^m = 0.$$

For the purpose of simplicity, let us define

$$(73)\quad \delta_i^n = \exp\left(\sum_m \gamma_i^m a_{mn}^i\right),$$

and

$$(74)\quad \varepsilon_j^m = \exp(-\gamma_j^m),$$

so that

$$(75)\quad \delta_i^n = \prod_m (\varepsilon_{ij}^m) - a_{mn}^i.$$

Then, eq. (72)′ can be written as

$$(76)\quad \varepsilon_j^m \sum_i \delta_i^n \exp(-u^m c_{ij}^m) - \sum_n a_{mn}^i \delta_i^n \sum_j \varepsilon_j^m \exp(-\mu^m c_{ij}^m) - y_i^m = 0,$$

which can be rearranged to give for ε_j^m:

$$(77)\quad \varepsilon_j^m = \frac{y_i^m + \sum_n a_{mn}^i S_i^n \sum_j \varepsilon_j^m \exp(-\mu^m c_{ij}^m)}{\sum_i \delta_i^n \exp(-\mu^m c_{ij}^m)}.$$

The equation cannot be solved explicitly for ε_j^m.[26] However, eq. (77) suggests an iterative-solution-procedure:

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26) Because ε_j^m still appears in the numerator of the right-hand side of eq. (77).

Iterative Procedures

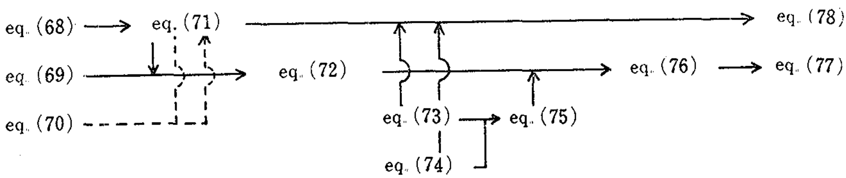
Firstly guess ε_j^m, then calculate δ_i^n from eq. (75), recalculate ε_j^m from eq. (77), and continue this cycle until the process converges. Then, using eqs. (73) and (74), eq. (71) for x_{ij}^m can be written as

(78) $x_{ij}^m = \delta_i^n \, \varepsilon_j^m \exp(-\mu^m c_{ij}^m)$.

Thus, in short, the entropy-maximizing model for what might be called the Leontief-Strout version of our case (1) is finally given by eqs. (78), (77), and (75).

The processes to reach our final results derived from the first-order conditions can be pictured as follows :

Figure 9
The Entropy-maximizing Model for the Leontief-Strout Version



V

In this paper we have discussed the theoretical and practical question of designing operational models based on interaction among different regions. We have shown how appropriate models of spatial systems can be derived by maximizing a function describing the entropy or information contained in such sytems subject to relevant constraints.

The theoretical result in this paper is that the models derived from entropy maximizing procedures are equivalent to many of the models already in use which have been derived empirically. The gravity model, among others, has been used as an empirical or phenomenological estimate for some years, and is in reasonable accord with reality.[27] These ideas shall be developed in relation to several urban and regional systems such as not only the interregional commodity flows, but also the transport, the location of population, etc.

However, if there is any desire to use the concept of entropy,then it should be made quite clear how it could be measured --- in terms of either objective probability or subjective probability.[28] This is the most crucial problem in entropy maximizing procedure.[29]

Application of the concept of entropy

We can summarize the types of application as follows :

1) hypothesis generation, or theory building,[30]
2) interpretation of theories.

The entropy maximizing procedure can be used to develop hypotheses. We can call this the main type of application. The general rule for generating hypotheses can be written as follows :

a) Set up the variables which define the system of interest, and write down the known constraint equations on these variables.
b) Define the entropy of the system, either directly or by using an associated probability distribution.
c) The variables can then be estimated by maximizing the entropy subject to the constraints.

In our example which was discussed in Section 4. 3, two types of constraint e-

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27) As for the gravity model, many refinments are possible. The Stouffer's intervening model may be seen as one of them.

28) The objective view is that probability is always capable of measurement by observation of frequency ratio in a random experiment. The subjective view regards probability as expressions of human ignorance; the probability of an event is merely a formal expression of our expectation that the event will, or did, occur based on whatever information is available.

29) It suggests us that if there is not any information in advance (*i. e.*, in the case of no constraint), the commodity flows tend to take on the uniform distribution (*i.e.*, the dispersing tendency). But, why? What kind of theoretical interpretation can be ready for this?

30) A theory is a well-tested hypothesis.

quations were specified in terms of x_{ij}^m. One was the cost constraint given by eq. (35), and the other was the input-output constraints given by eq. (1)′, which signify the different production structures with respect to each region. We then defined the entropy of the probability distribution associated with x_{ij}^m as $-\sum_i \sum_j \sum_m x_{ij}^m \, l_n \, x_{ij}^m$ given by eq. (66). This is our objective function which should be maximized. Finally, from the method of the Lagrangean multipliers, we obtained the optimum values of x_{ij}^m as eq. (78). Their derivation was schematically pictured in Figure 9.

Many, if not most, hypotheses thus generated could be produced by more conventional means. However, at the very least, the entropy maximizing procedure enables us to handle extremely complex situations in a consistent way. In fact, past experience has shown that this sort of consistency is very difficult to achive otherwise. In this sense, the entropy maximizing procedure can be well regarded as more significant and meaningful approach among others.[31)]

When we construct hypotheses or models, it is often necessary to include terms which are difficult to interpret in any direct way. These are high-level theoretical concepts which are often well removed from possibilities of direct measurement. They may be the parameters of a model, such as μ^m in the Leontief-Strout version of our case (1).

Suppose the model given by eqs. (75), (77) and (78) could be developed and used fruitfully without entropy maximizing procedures. It is then possible to write down the set of constraint equations which give rise to the same model, in this example eqs. (1′) and (35). The parameter μ^m, for example, is then seen to be the Lagrangean multiplier associated with the constraint eq. (35), and the interpretaton of this equation adds to our knowledge of the role μ^m plays in the model.

The sign of μ^{m*} tells us the direction of the change in the optimum

31) However, it should be emphasized that the hypotheses which are generated should be tested in the same way as hypotheses generated by any other procedure.

value of $S(X)$ given by eq. (66).[32] A positive μ^{m*} means that if the right-hand side of the constraint, eq. (35), increases, so does $S(X^*)$: $\mu^{m*} < 0$ means that an increase in the constraint constant C^m is accompanied by a decrease in $S(X^*)$. In fact, the value of μ^{m*} represents the partial derivative of $S(X)$, evaluated at X^* with respect to the right-hand side of the constraint, C^m.[33]

In this sense, although the Lagrangean multipliers (say μ^m) are not the variables whose optimum values are of direct interest in the problem, it does turn out that those multipliers provide useful information about the constraints. Keeping this fact in our mind, we can introduce any additional constraint --- which might be expected to cause the significant change in the optimum solution --- to our model and then we can also evaluate how effectively such a hypothetical constraint works in the entropy maximizing procedures.[34]

Necessity for the model building

Urban and regional models are of interest for two main reasons :

1) Model building is at the root of all scientific study, and urban and regional modelling is part of an attempt to achieve a scientific understanding of cities and regions.

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32) Asterisk denotes that μ^m is evaluated at the point that satisfies the first-order conditions given by eqs. (68), (69) and (70).

33) Recall that the form of the constraint used in the Lagrangean form is $C^m - \sum_i \sum_j x_{ij}^m c_{ij}^m = 0$. From eq. (67), at X^* and μ^{m*}, $\partial L^*/\partial C^m = \mu^{m*}$. Since at optimum the constraint (1)′ and (35) must hold (*i. e.*, eqs. (69) and (70) must hold), we obtain $L^* = S(X^*)$, and hence $\partial S(X^*)/\partial C^{m*} = \mu^{m*}$.

34) Note that the discussion on the interpretation of the Lagrangean multiplier can be applied to the case where the problem has some inequality constraints. For example, the rule for the problem ; maximize $S(X)$ subject to $g(X) \leq C$, can be written as follows : Solve the problem as if the constraint were an equality using both first-and second-order conditions ; then (1) if $\mu^* \geq 0$, the maximum is on the boundary, therefore X^* found by assuming $g(X) = C$ is correct, (2) if $\mu^* < 0$, the maximum is interior to the boundary, therefore redo the problem ignoring the constraint completely.

2) A variety of severe urban and regional problems exist, and associated planning activity has become increasingly important; urban and regional modelling is a part of the advance on this front.

The most crucial problem in the model building lies in the determination of relationships among variables, which is often called the model-specification. It is dependent on an exogenously given objective, since the model must always be built in order to meet some a priori objective. Such a objective usually includes description, prediction and/or impact analysis based on simulation.

In Japan, we have been suffering from severe regional problems, such as pollution and congestion. Quite recently, in order to remedy those problems, Prime Minister K. Tanaka presented an ambitious decentralization plan known as "A Plan for Remodelling the Japanese Archipelago." To make his decentralization plans work requires a vast improvement in air, rail and road transport. By 1985, he insists, there must be an additional 6,000 miles of railway lines and a series of high-speed trains crisscrossing the archipelago. By then, the islands will have been connected by the bridges and tunnels.

If all goes according to that plan, 32 new expressways will also be built by 1985, and the new travel network will enable a person to journey to any point in Japan within one day. As might be expected, Tanaka's plans have already evoked a considerable amount of criticism. To conservative, they are too visionary. The left charges they ignore basic social inequalities. However, no matter how we like it or not, we are facing a number of severe theoretical problems around it. Our study is often a multidisciplinary one in the sense that we need to use concepts from several disciplines --- economics, geography, sociology, etc.

The concept of entropy has, until recently, been used primarily in the nonsocial sciences. From our discussion in this paper, it turned out that the entropy maximizing procedure has a useful and valuable role in one branch of the social sciences --- the study of interregional commodity flows. When we take account of data availability, processing cost, and time relevance, the entropy maximizing procedure enables us to tackle some of our basic problems, such as an impact analysis, in a fruitful way. It endeavors to shape tools that can help assess and anticipate impact estimates stemming from the new decentralization plan previously stated.

It should be recalled that some problems still remain to be solved in order to apply the entropy maximizing procedure to the real-world situations: both how to estimate the transport cost of a unit of commodity m (*i. e.*, c_{ij}^{m}), and what sort of theoretical implications can be given to the maximizing motivation of our objective function $S(X)$.

Some of them may be solved in the process of empirical implementations.[35] But, some may not, and require further theoretical examinations. However, any model should be evaluated not only from the theoretical viewpoint, but also from the empirical or practical viewpoint. Thus, although an entropy maximizing procedure has some deficiencies in the sense that it is hard to give the theoretical interpretations to the maximizing motivation, so long as it gives good predictions by reproducing complex real-world situations, it must be favorably evaluated because of its flexibility and operationality.

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35) For the detailed discussion about the estimation of Japanese 1963 interregional trade flows, see Polenske (8).

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