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# Plane Algebraic Curves Drawn by a Spirograph and the Isogonal Conjugate for a Triangle

by

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## Abstract

Arbitrary curves drawn by a spirograph can be obtained as loci of the isogonal conjugate to a point for an equilateral triangle which moves along a circle.

## § 1. Introduction

In the sequel to [**4**]-[**9**], [**16**]-[**17**] our study aims to develop a drawing tool on a display for experimental research on various curves using computers.

In elementary geometry we have five signifcant notions for a triangle ; that is, the center of gravity, the center of an inscribed circle, the center of an escribed circle, the circumcenter and the orthocenter of a triangle. As a similar notion we have the isogonal conjugate to a point for a triangle.

Which curve is drawn as a locus of the isogonal conjugate to a point for a triangle which moves under a certain condition? In this study we limit ourselves to the case where a triangle is fixed while the point moves along a distinguished curve. Then our main concern is to find various curves with simple algebraic expressions as a locus of the isogonal conjugate to a point for a triangle.

Watching the display, the first author noticed that some of them were quite similar to ones drawn by a spirograph. Then we have a following question :

Can a curve drawn by a spirograph be obtained by a locus of an isogonal conjugate to a point for an equilateral triangle which moves along a simple curve ?

In this paper we shall give an affirmative answer to it.

For terminology of geometry throughout the paper, consult [**2**], [**3**], [**11**], [**15**], [**18**] and [**19**].

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## § 2. A program for drawing a locus

On the plane let us consider a given triangle $\triangle ABC$ and a point $P$ different from the vertices of the triangle (**Fig. 1**).

Reflect the lines connecting the vertices of $\triangle ABC$ with $P$ in the bisectors of the corresponding angles of the triangle. It turns out that three lines always intersect in a single point (or are parallel, i.e., intersects in the point of infinity $\infty$), which we denote $P'$. The point $P'$ is called the *isogonal conjugate* of $P$ for $\triangle ABC$; the map $\varphi : \mathbf{R}^2 \setminus \{A, B, C\} \longrightarrow \mathbf{R}^2 \cup \{\infty\}$ defind by $\varphi(P) = P'$ is called the *isogonal conjugation* for $\triangle ABC$.

**Proposition.** *For any point $P$ different from the vertices of $\triangle ABC$, it lies on the circumcircle of $\triangle ABC$ if and only if its isogonal conjugate is the point of infinity; that is, $\varphi(P) = \infty$.*

Let $A(0, 1)$, $B\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $C\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ be the vertices of an equilateral triangle, and $P(u,v)$ a point diffierent from the vertices. The point $P'(x, y)$ is determined as the intersection of two straight lines $P'B$ and $P'C$ given by

$$(*) \quad \begin{cases} P'B \; : \quad y = m_b\left(x + \dfrac{\sqrt{3}}{2}\right) - \dfrac{1}{2} \\ \\ P'C \; : \quad y = m_c\left(x - \dfrac{\sqrt{3}}{2}\right) - \dfrac{1}{2}, \end{cases}$$

where $$m_b = \frac{\sqrt{3}u - v + 1}{u + \sqrt{3}v + \sqrt{3}},$$

$$m_c = \frac{-\sqrt{3}u - v + 1}{u - \sqrt{3}v - \sqrt{3}}.$$

Which curve is drawn as a locus of the isogonal conjugate $P'$ to a point $P$ for $\triangle ABC$ which moves along a distinguished curve $\mathscr{C}$?

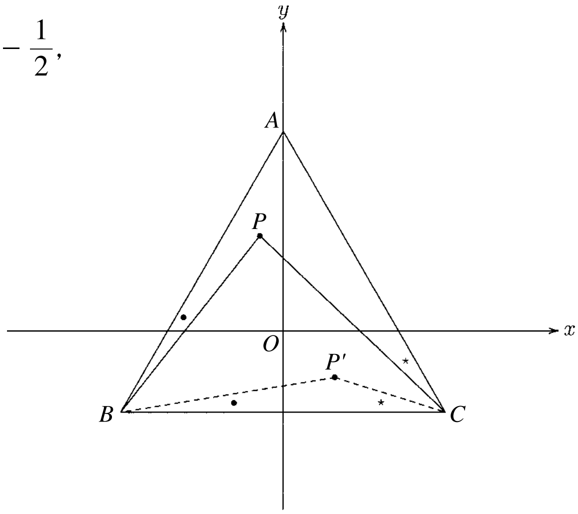

**Fig. 1**

We divide our operation of a drawing tool into the drawing part and the printing part, due to the circumstances of our computer machines.

PART ONE : To draw a locus of the isogonal conjugate to a point for a given triangle.

A program of our drawing game for **Case 1** of Example 1 in § 3 is written in Visual Basic Ver. 6.0 by Microsoft Corporation and consists of the following four steps (see **List 1**).

**Step 1.** Set the coordinates axes and $\triangle ABC$ in black.
**Step 2.** Set the initial position of a point $P$ on curve $\mathscr{C}$ in blue.
**Step 3.** Plot the isogonal conjugate $P'$ in red.
**Step 4.** When one moves the point $P$ continuously along the curve $\mathscr{C}$ by the mouse, the point $P'$ continuously draws a locus $\mathscr{L}$ with a solid line in red.

PART TWO : To process a bitmap file (bmp) by pLaTeX2$_\varepsilon$, to exhibit it on another display, and to print it.

OUTLINE of the PROGRAM. Let $\triangle ABC$ be the equilateral triangle as above and $\mathscr{C}$ : $x^2+y^2 = 4$. Where a point $P(u, v)$ on $\mathscr{C}$ is selected by the mouse, then the isogonal conjugate $P'(x, y)$ to $P$ is determined by the equations (∗).

The program for drawing a locus $\mathscr{L}$ of the point $P'$ is given in **List 1**. In this case we will have the curve

$$\mathscr{L} :\ 3x^4+6x^2y^2+3y^4-6x^2y+2y^3-9x^2-9y^2+4=0$$

on a display (**Fig. 2**).

We keep the same notations as in this section throughout the paper.

As a locus of the isogonal conjugate to a point for a given equilateral triangle $\triangle ABC$ we have various algebraic curves.

**Example 1.** *Each of the following algebraic curves* $\mathscr{L}$ *can be obtained as a locus of the isogonal conjugate* $P'$ *to a point* $P$ *for the equilateral triangle* $\triangle ABC$ *which moves along the curve* $\mathscr{C}$.

**Case 1** (**Fig. 2**). $\mathscr{C}: x^2+y^2=4$, *the circle.*

$$\mathscr{L}: 3x^4 + 6x^2y^2 + 3y^4 - 6x^2y + 2y^3 - 9x^2 - 9y^2 + 4 = 0.$$

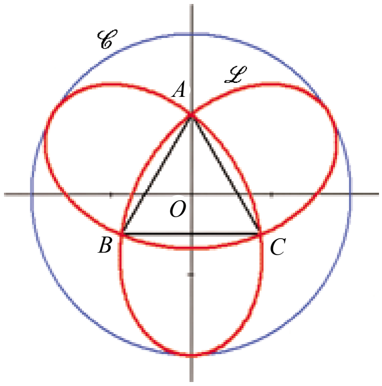


**Fig. 2**

**Case 2** (**Fig. 3**). $\mathscr{C}: x^2+y^2=\dfrac{1}{4}$, *the circle.*

$$\mathscr{L}: 3x^4 + 6x^2y^2 + 3y^4 + 24x^2y - 8y^3 + 6x^2 + 6y^2 - 1 = 0.$$

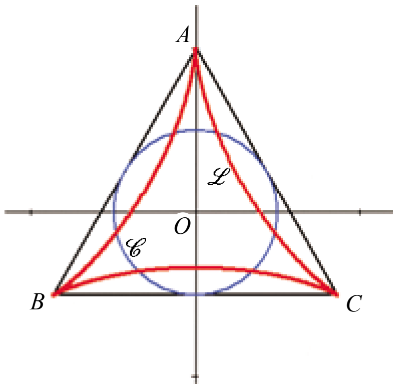


**Fig. 3**

*Proof.* Let $P(u, v)$ and $P'(x, y)$.

**Case 1.** Find the Groebner bases (see [**1**], [**12**]) by applying Mathematica Ver 6.0 by Wolfram Research Inc., to the following code :

```
f1 := u^2 + v^2 - 4
f2 := (u + Sqrt[3]*v + Sqrt[3])*mb - (Sqrt[3]*u - v + 1)
f3 := (u - Sqrt[3]*v - Sqrt[3])*mc - (-Sqrt[3]*u - v + 1)
f4 := (2*y + 1) - mb*(2*x + Sqrt[3])
f5 := (2*y + 1) - mc*(2*x - Sqrt[3])
GroebnerBasis[{f1, f2, f3, f4, f5}, {u, v, mb, mc, x, y}]
```

Then we obtain the term

$$4-9x^2+3x^4-6x^2y-9y^2+6x^2y^2+2y^3+3y^4$$

in the list of the generating Groebner bases. Therefore, we have

$$3x^4+6x^2y^2+3y^4-6x^2y+2y^3-9x^2-9y^2+4=0$$

as an equation of $\mathscr{L}$.

The proof for case 2 is omitted. □

## § 3. Results

A spirograph is a geometric drawing toy that produces various curves known as hypotrochoids and epitrochoids.

Let the fixed circle have radius $a$ and have its center at the origin $O$ of an $xy$-coordinate system (**Fig. 4**). Let the rolling circle have radius $r\,(r>0)$ and have center $P$ which travels with it, and let $\theta$ be the angle which $\overrightarrow{OP}$ makes with the $y$-axis so that

$$\overrightarrow{OP} = (a-r)(-\sin\theta,\ \cos\theta).$$

We suppose that $\theta>0$, so the spoke $PR$ in the moving wheel rotates in a positive direction, while the wheel (the circle) $P$ contacts with the circle $O$ at $S$, and $Q$ the point on the spoke (or the extended spoke) $PR$. For $\theta=0$, denote the initial position of the points $P$, $Q$, $R$ by $P_0$, $Q_0$, $R_0$, respectively. Let $Q_0=(0, q_0)$ and $t=a-q_0\neq r$.

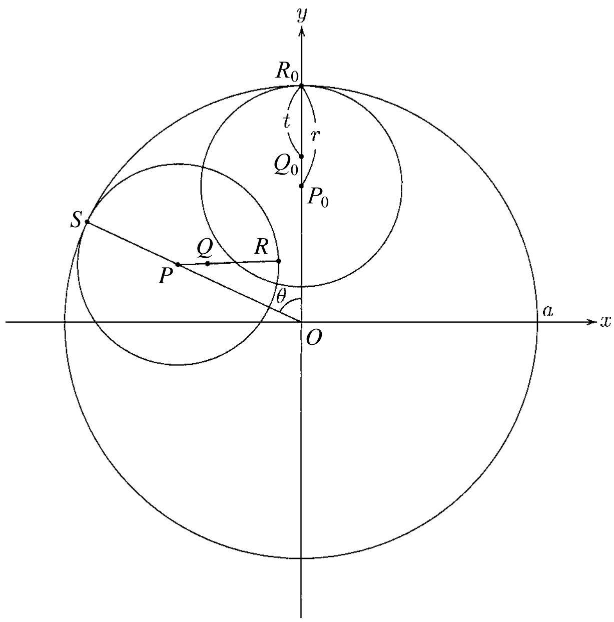


Fig. 4

We have the following :

**Theorem 1.** *Parametric equations for the locus* $\mathscr{K}$ *of the point* $Q$ *are given by*

$$x = -(a-r)\sin\theta - (r-t)\sin\left(\frac{r-a}{r}\right)\theta,$$

$$y = (a-r)\cos\theta + (r-t)\cos\left(\frac{r-a}{r}\right)\theta.$$

*Proof.* Let $\alpha = \angle RPS$. Since $\overset{\frown}{SR} = \overset{\frown}{SR_0}$, we have

$$r\alpha = a\theta, \qquad \text{or} \qquad \alpha = \frac{a}{r}\theta.$$

Since the angle which $\overrightarrow{PR}$ makes with the $x$-axis equals $\frac{\pi}{2} + \theta - \alpha$, we have

$$\overrightarrow{PQ} = (r-t)\left(-\sin\left(\frac{r-a}{r}\right)\theta,\ \cos\left(\frac{r-a}{r}\right)\theta\right).$$

Therefore, we have the conclusion because $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$. □

**Example 2.** *Each of the following algebraic curves $\mathscr{K}$ can be obtained as a locus of the point $Q$ in Theorem 1.*

**Case 1** (**Fig. 5**). $a=1,\ r=\frac{1}{3},\ t=\frac{5}{3}.$

$$\mathscr{K}:\ 3x^4+6x^2y^2+3y^4-6x^2y+2y^3-9x^2-9y^2+4=0.$$

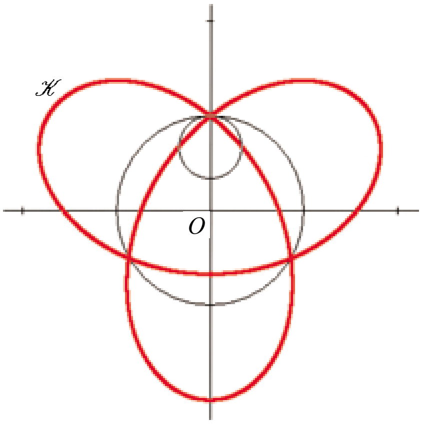


**Fig. 5**

**Case 2** (**Fig. 6**). $a=1,\ r=\frac{1}{3},\ t=0.$

$$\mathscr{K}:\ 3x^4+6x^2y^2+3y^4+24x^2y-8y^3+6x^2+6y^2-1=0.$$

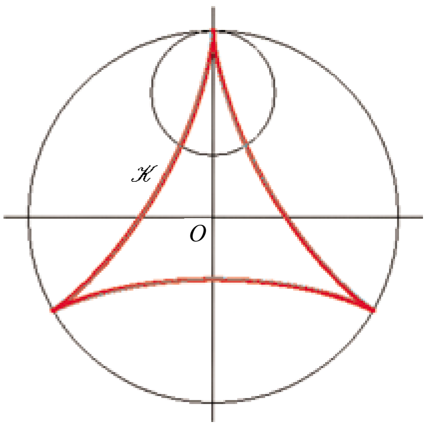


**Fig. 6**

Proof.

**Case 1.** In this case, we have

$$x=\frac{1}{3}(-2\sin\theta-4\sin 2\theta),$$
$$y=\frac{1}{3}(2\cos\theta-4\cos 2\theta).$$

Find the Groebner bases by applying Mathematica to the following code :

```
f1 := c^2 + s^2 - 1
c2 := 2*c^2 - 1
s2 := 2*c*s
f2 := x - (-2*s - 4*s2)/3
f3 := y - (2*c - 4*c2)/3
GroebnerBasis[{f1, f2, f3}, {s, c, y, x}]
```

Then we obtain the term

$$4-9x^2+3x^4-6x^2y-9y^2+6x^2y^2+2y^3+3y^4$$

in the list of the generating Groebner bases. Therefore, we have

$$3x^4+6x^2y^2+3y^4-6x^2y+2y^3-9x^2-9y^2+4=0.$$

as an equation of $\mathscr{K}$.

The proof for Case 2 is omitted. □

**Theorem 2.** *The algebraic curve*

$$\mathscr{L}:\ (b^2-1)x^4+2(b^2-1)x^2y^2+(b^2-1)y^4-6x^2y+2y^3-(2b^2+1)x^2-(2b^2+1)y^2+b^2=0$$

*can be obtained as a locus of the isogonal conjugate* $P'$ *to a point* $P$ *for the equilateral triangle* $\triangle ABC$ *which moves along the circle*

$$\mathscr{C}:\ x^2+y^2=b^2 \qquad (b>0,\quad b\neq 1).$$

*Proof.* Let $P(u, v)$ and $P'(x, y)$. Find the Groebner bases by applying Mathematica to the following code :

```
f1 := u^2 + v^2 - b^2
f2 := (u + Sqrt[3]*v + Sqrt[3])*mb - (Sqrt[3]*u - v + 1)
f3 := (u - Sqrt[3]*v - Sqrt[3])*mc - (-Sqrt[3]*u - v + 1)
f4 := (2*y + 1) - mb*(2*x + Sqrt[3])
f5 := (2*y + 1) - mc*(2*x - Sqrt[3])
GroebnerBasis[{f1, f2, f3, f4, f5}, {u, v, mb, mc, x, y}]
```

Then we obtain the term

$$-b^2+(1+2b^2)x^2+(1-b^2)x^4+6x^2y+(1+2b^2)y^2+(2-2b^2)x^2y^2-2y^3+(1-b^2)y^4$$

in the list of the generating Groebner bases. Therefore, we have

$$(b^2-1)x^4+2(b^2-1)x^2y^2+(b^2-1)y^4-6x^2y+2y^3-(2b^2+1)x^2-(2b^2+1)y^2+b^2=0$$

as an equation of $\mathscr{L}$. □

**Theorem 3.** *Let* $f(x, y)$, $g(x, y)$ *be polynomials that define the curves* $\mathscr{K}$ *in Theorem 1,* $\mathscr{L}$ *in Theorem 2, respectively ; that is,*

$$
\begin{aligned}
f(x, y) &= -(r-t)^2x^4 - 2(r-t)^2x^2y^2 - (r-t)^2y^4 - 24r^2(r-t)x^2y \\
&\quad + 8r^2(r-t)y^3 - 2(3r-t)(r+t)(3r^2-2rt+t^2)x^2 \\
&\quad - 2(3r-t)(r+t)(3r^2-2rt+t^2)y^2 + (3r-t)^3(r+t)^3 \qquad (r \neq t),
\end{aligned}
$$

$$
\begin{aligned}
g(x, y) &= (b^2-1)x^4 + 2(b^2-1)x^2y^2 + (b^2-1)y^4 - 6x^2y + 2y^3 \\
&\quad - (2b^2+1)x^2 - (2b^2+1)y^2 + b^2 \qquad (b>0, \quad b \neq 1).
\end{aligned}
$$

*If both curves* $f(x, y)=0$ *and* $g(x, y)=0$ *coincide, then it holds that*

$$
r = \left| \frac{b}{2(b^2-1)} \right|, \qquad t = \frac{b(2b \pm 1)}{2(b^2-1)}.
$$

*Proof.* Put $f(x, y) = kg(x, y)$. By comparing the coefficients of both sides we have

$$
\begin{aligned}
-(r-t)^2 &= k(b^2-1) \\
(3r-t)^3(r+t)^3 &= kb^2 \\
8r^2(r-t) &= 2k \\
-2(3r-t)(r+t)(3r^2-2rt+t^2) &= k(-2b^2-1).
\end{aligned}
$$

Find the Groebner bases by applying Mathematica to the following code :

```
f1 := -(r - t)^2 - k*(b^2 - 1)
f2 := (3*r - t)^3*(r + t)^3 - k*b^2
f3 := 8*r^2*(r - t) - k*2
f4 := -2*(3*r - t)*(r + t)*(3*r^2 - 2*r*t + t^2) - k*(-2*b^2 - 1)
GroebnerBasis[{f1, f2, f3, f4}, {t, k, b, r}]
Factor[%]
```

Then we obtain the term

$$
r^4(-1+3r)^2(1+3r)^2(-b-2r+2b^2r)(b-2r+2b^2r)
$$

in the list of the generating Groebner bases. By $r>0$, we have

$$
(-1+3r)^2(-b-2r+2b^2r)(b-2r+2b^2r) = 0. \tag{1}
$$

On the other hand, find the Groebner bases by applying Mathematica to the following code :

```
f1 := -(r - t)^2 - k*(b^2 - 1)
f2 := (3*r - t)^3*(r + t)^3 - k*b^2
f3 := 8*r^2*(r - t) - k*2
f4 := -2*(3*r - t)*(r + t)*(3*r^2 - 2*r*t + t^2) - k*(-2*b^2 - 1)
GroebnerBasis[{f1, f2, f3, f4}, {r, k, b, t}]
Factor[%]
```

Then we obtain the term

$$t^4(2+3t)^2(-b-2b^2-2t+2b^2t)(b-2b^2-2t+2b^2t)$$

in the list of the generating Groebner bases.
Hence, we have

$$t^4(2+3t)^2(-b-2b^2-2t+2b^2t)(b-2b^2-2t+2b^2t)=0. \qquad (2)$$

By (1), (2), we have the following :
If $-1+3r\neq 0$ and $t(2+3t)\neq 0$, then

$$r=\left|\frac{b}{2(b^2-1)}\right|, \qquad t=\frac{b(2b\pm 1)}{2(b^2-1)}.$$

The results also hold for the case $-1+3r=0$ or $t(2+3t)=0$. □

As an application to Theorem 3 we have the following :

**Example 3** (**Fig. 7**). *The algebraic curve*

$$\mathscr{L}: \ 7x^4+14x^2y^2+7y^4+96x^2y-32y^3+34x^2+34y^2-9=0$$

*can be obtained as a locus of the isogonal conjugate* $P'$ *to a point* $P$ *for the equilateral triangle* $\triangle ABC$ *which moves along the circle*

$$\mathscr{C}: \ x^2+y^2=\frac{9}{16}.$$

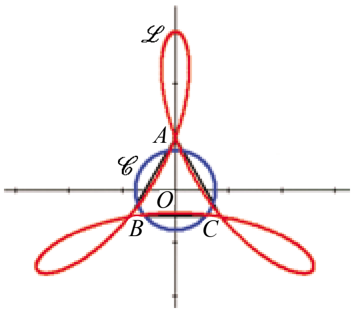


**Fig. 7**

**Example 4** (**Fig. 8**). *For* $b=\dfrac{3}{4}$, $r=\dfrac{6}{7}$, $t=-\dfrac{3}{7}$, *the locus of the point* $Q$ *in Theorem 1 is given by*

$$\mathscr{K}: \ 7x^4+14x^2y^2+7y^4+96x^2y-32y^3+34x^2+34y^2-9=0,$$

*which coincides to the curve* $\mathscr{L}$ *in Example 3.*

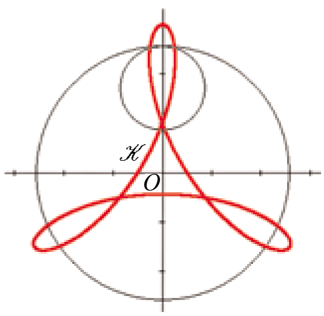


**Fig. 8**

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**List 1. A program for drawing a locus**

```
'-----------------------------------------------
'      Specifying statements of the general variables
'-----------------------------------------------
Dim sx, sy, ex, ey As Double
Dim ax, ay, bx, by, cx, cy As Double
Dim pi As Double
Dim tb, tc As Double
Dim k As Integer
Dim j As Double
Dim qr As Double

'------------------------------
'      Drawing the initial figure
'------------------------------
Private Sub Form_Activate()
  Dim r, X, Y As Double

  Form1.AutoRedraw = True
  Form1.Line (sx, 0)-(ex, 0)
  Form1.Line (0, sy)-(0, ey)

  sxn = Int(sx): syn = Int(sy)
  exn = Int(ex): eyn = Int(ey)

  h = 0.04
  For i = sxn To exn
    Form1.Line (i, -h)-(i, h)
  Next i

  For i = syn To eyn
    Form1.Line (-h, i)-(h, i)
  Next i

  Form1.DrawWidth = 2
  Form1.Line (ax, ay)-(bx, by), vbBlack
  Form1.Line (cx, cy)-(ax, ay), vbBlack
  Form1.Line (bx, by)-(cx, cy), vbBlack

  Form1.AutoRedraw = False

  Form1.Circle (0, 0), 1.5, vbBlue

  tb = Agl(ax, ay, bx, by, bx + 1, by)
  tc = Agl(ax, ay, cx, cy, cx + 1, cy)

  k = 1
  Form1.DrawWidth = 4
  j = 0
  Equ_Agl 1.5
  qr = 1.5
  Text1.Text = qr
End Sub
```

```
'------------------------------------------------------
'     Setting the form, the coordinate axes and the initial
'     values of the coordinates of each vertex of a triangle
'------------------------------------------------------
Private Sub Form_Load()
  Form1.Top = 50
  Form1.Left = 500
  Form1.Height = 0.95 * Screen.Height
  Form1.Width = 0.7 * Screen.Width
  sx = -4.2
  ex = 4.2
  wx = ex - sx
  wy = wx * Form1.ScaleHeight / Form1.ScaleWidth
  sy = -0.5 * wy
  ey = 0.5 * wy
  Form1.Scale (sx, ey)-(ex, sy)
  Form1.BackColor = vbWhite
  Form1.DrawWidth = 1
  pi = 4 * Atn(1)

  ax = 0: ay = 1
  bx = -Sqr(3) / 2: by = -0.5
  cx = Sqr(3) / 2: cy = -0.5

  Text1.FontSize = 16
End Sub

'---------------------------------------------------
'     Action corresponding to the left button of mouse
'---------------------------------------------------
Private Sub Form_MouseDown(Button As Integer, Shift As Integer, X As Single, Y As Single)
  If k = 2 Then
    k = 1
    Exit Sub
  End If

  Form1.Cls
  Form1.DrawWidth = 2
  r = Sqr(X ^ 2 + Y ^ 2)
  Form1.Circle (0, 0), r, vbBlue
  Form1.DrawWidth = 4

  Equ_Agl r
  qr = r
  Text1.Text = qr
  k = 2
End Sub
```

```
'---------------------------------------------------
'     Action corresponding to the movement of mouse
'---------------------------------------------------
Private Sub Form_MouseMove(Button As Integer, Shift As Integer, X As Single, Y As Single)
  If k = 1 Then Exit Sub

  Form1.Cls

  Form1.DrawWidth = 2
  r = Sqr(X ^ 2 + Y ^ 2)
  Form1.Circle (0, 0), r, vbBlue
  Form1.DrawWidth = 4

  Equ_Agl r
  qr = r
  Text1.Text = qr
End Sub

'-------------------------------------
'     Calculating the value of an angle
'-------------------------------------
Private Function Agl(xa, ya, xb, yb, xc, yc)
  a = Sqr((xc - xb) ^ 2 + (yc - yb) ^ 2)
  b = Sqr((xa - xc) ^ 2 + (ya - yc) ^ 2)
  c = Sqr((xb - xa) ^ 2 + (yb - ya) ^ 2)

  If a < 0.0000001 Or c < 0.0000001 Then Exit Function

  X = (a ^ 2 + c ^ 2 - b ^ 2) / (2 * a * c)
  If 1 - X ^ 2 < 0.0000001 Then
    If X > 0 Then
      t = 0
    Else
      t = pi
    End If
  Else
    t = pi / 2 - Atn(X / Sqr(1 - X ^ 2))
  End If

  h = xa * yb + xb * yc + xc * ya - xa * yc - xb * ya - xc * yb
  If h < 0 Then t = -t

  Agl = t
End Function
```

```
'-----------------------------------------
'      Presenting of the isogonal conjugate
'-----------------------------------------
Private Sub Disp(px, py)
  sb = Agl(px, py, bx, by, cx, cy)
  mb = Tan(tb - sb)
  sc = Agl(px, py, cx, cy, bx, by)
  mc = Tan(tc - sc)
  qx = (mb * bx - mc * cx - by + cy) / (mb - mc)
  qy = mb * (qx - bx) + by
  If Abs(qx) > ex Or Abs(qy) > ey Then Exit Sub

  Form1.PSet (qx, qy), QBColor(12)
End Sub

Private Sub Equ_Agl(r)
  For t = 0 To 2 * pi Step 0.01
    X = r * Cos(t)
    Y = r * Sin(t)

    Disp X, Y
  Next t
End Sub

Private Sub Ext_Click()
  For t = 0 To 2 * pi Step 0.0001
    X = qr * Cos(t)
    Y = qr * Sin(t)

    Disp X, Y
  Next t
End Sub

Private Sub Text1_DblClick()
  r = Val(Text1.Text)
  Form1.Cls

  Form1.DrawWidth = 2
  Form1.Circle (0, 0), r, vbBlue
  Form1.DrawWidth = 4

  Equ_Agl r
  qr = r
  k = 1
End Sub
```