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On the Tensor Product of Field Extensions

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INTRODUCTION. All rings considered in this paper are assumed to be commutative with identity, and all ring homomorphisms are unital. We use $Q(A)$ to denote the quotient field of an integral domain A. Let $A \subseteq B$ be integral domains, trans.deg$_A B$ donotes the transcendence degree of the quotient field of B over the quotient field of A.

Let $K_1, \cdots, K_n$ be fields of finite transcendence degree $t_1, \cdots, t_n$ over a field k. Sharp and Vamos [8] showed that the dimension of the tensor product $K_1 \otimes_k \cdots \otimes_k K_n$ is $t_1 + \cdots + t_n - \max_i\{t_i\}$. O'Carroll and Qureshi [7] conjectured that $K_1 \otimes_k \cdots \otimes_k K_n$ is an equidimensional Hilbert ring, and proved the conjecture in special case. Trung [9], Howie and O'Carroll [4] and Nagata [6] proved the conjecture. Lee and Nam [5] generalized the O'Carroll and Qureshi's result. In this paper, as an application of Lemma 1 and Theorem 2 below, we prove that the tensor product $K_1 \otimes_k \cdots \otimes_k K_n$ is an equidimensinal Hilbert ring.

LEMMA 1. Let $B = k[x_1, \cdots, x_m]$ be a finitely generated integral domain over an infinite field k and let $A_1, \cdots, A_n$ be subrings of B containing k. Set $S_i = A_i - \{0\}$ $(i = 1, \cdots, n)$. Let S be the multiplicatively closed set generated by $\cup_{i=1}^{n} S_i$. Suppose thet $\dim B > \max_i\{\dim A_i\}$. Then

(1) there exist elements $y, y_{i2}, \cdots, y_{im_i}$ $(1 \leqq i \leqq n)$ of B satisfying

  (i) $S_i^{-1}B$ is integral over $Q(A_i)[y, y_{i2}, \cdots, y_{im_i}]$.

  (ii) $y, y_{i2}, \cdots, y_{im_i}$ are algebraically independent over $Q(A_i)$.

(2) There exist infinitely many height one prime ideals of $S^{-1}B$.

PROOF. (1) [6, p.376 (3)] We have $\dim A_i = \text{trans.deg}_k A_i$ by [3, Proposition (2.3), (b)], hence $\text{trans.deg}_k B = \dim B > \max_i\{\text{trans.deg}_k A_i\}$. Since k is infinite, by the normalization theorem ([1, p.69, exercise 16], [2, p.84, exercise 3]), there are algebraically independent elements $y_{i1}, \cdots, y_{im_i}$ over $Q(A_i)$ such that $S_i^{-1}B = Q(A_i)[x_1, \cdots, x_m]$ is integral over $Q(A_i)[y_{i1}, \cdots, y_{im_i}]$, where each y_{ij} is a linear combinations of $x_1, \cdots, x_m$ with coefficients in k. Since y_{i1} is obtained as any element of $\{c_1x_1 + \cdots + c_mx_m \mid (c_1, \cdots, c_m) \in k^m - V_i\}$ where V_i is a suitable Zariski closed subset of k^m. Since $\cup_i V_i$ is a proper subset of k^m, taking $(c_1, \cdots, c_m)$ in $k^m - \cup_i V_i$, we can choose $y = y_{11} = \cdots = y_{n1} = c_1x_1 +$

$\cdots + c_m x_m$.
(2) Let λ be any element of k, and let P be a minimal prime ideal of $(y-\lambda)B$. Set $B_i = A_i[y, y_{i2}, \cdots, y_{im_i}]$, $p = P \cap B_i$. Since B is an affine domain over k, trans.$\deg_k B$ = trans. $\deg_k B/P + 1$ holds. Now trans.$\deg_k B$ = trans.$\deg_k B_i$ implies that trans. $\deg_k B/P \leqq$ trans. $\deg_k B_i/p$. Moreover trans.$\deg_k B_i/p$ + height$(p) \leqq$ trans.$\deg_k B_i$. So we see that height$(p) \leqq 1$, and hence p is of height one. Since $(y-\lambda)A_i[y, y_{i2}, \cdots, y_{im_i}]$ is a prime ideal of $A_i[y, y_{i2}, \cdots, y_{im_i}]$, we have $p = (y-\lambda)A_i[y, y_{i2}, \cdots, y_{im_i}]$, so $P \cap S_i = \emptyset$. Therfefore $P \cap S = \emptyset$, so $S^{-1}P$ is a height one prime ideal of $S^{-1}B$ containing $y-\lambda$. Since k is an infinite field, there exist infinitely many height one prime ideals of $S^{-1}B$.

THEOREM 2. Let B be a finitely generated integral domain over a field k and let $A_1, \cdots, A_n$ be subrings of B containing k. Set $S_i = A_i - \{0\}$ $(i = 1, \cdots, n)$. Let S be the multiplicatively closed set generated by $\cup_{i=1}^n S_i$. Then

(1) $S^{-1}B$ is a Hilbert ring.
(2) Every maximal saturated chain of prime ideals in $S^{-1}B$ has same length $\dim B - \max_i\{\dim A_i\}$.

PROOF. Let $\bar{k}$ be the algebraic closure of k. By considering $B \otimes_k \bar{k}$ and $A_i \otimes_k \bar{k}$ $(1 \leqq i \leqq n)$ if necessary, we may asuume that k is infinite. Let $m = \dim D - \max_i\{\dim A_i\}$. We prove two statements by induction on m. If $m = 0$ then $\dim B = \dim A_i$ for some i. Then $S^{-1}B$ is algebraic over $Q(A_i)$, so $S^{-1}B$ is a field. Thus the case $m = 0$, the assertions (1), (2) is clear. Suppose $m \geqq 1$. Let $0 \subset P_1 \subset \cdots \subset P_r$ be a maximal saturated chain of prime ideals in $S^{-1}B$. Set $p_1 = P_1 \cap B$, $B = B/p_1$, $\overline{S} = \varphi(S)$, $\bar{A}_i = \varphi(A_i)$ $(1 \leqq i \leqq n)$ where $\varphi : B \longrightarrow \overline{B}$ is a canonical ring homomorphism. Then $A_i \cong \bar{A}_i$, $\dim \overline{B} = \dim B - 1$ and $\overline{S}$ is the multiplicatively closed set generated by $\cup_{i=1}^n(\bar{A}_i - \{0\})$. By induction $S^{-1}B/P_1 \cong (\overline{S})^{-1}\overline{B}$ is a Hilbert domain, and every maximal saturated chain of prime ideals in $S^{-1}B/P_1$ has length $m-1$. Since $0 \subset P_2/P_1 \subset \cdots \subset P_r/P_1$ is a maximal saturated chain of prime ideals in $S^{-1}B/P_1$, we have $r-1 = m-1$, so $r = m$. Thus the statement (2) holds. $S^{-1}B/P$ is a Hilbert domain for each height one prime ideal P of $S^{-1}B$, and a noetherian domain $S^{-1}B$ has infinitely many height one prime ideals by Lemma 1, (2). Therefore $S^{-1}B$ is a Hilbert ring.

COROLLARY 3. Let $K_1, \cdots, K_n$ be fields of finite transcendence degree $t_1, \cdots, t_n$ over a field k $(n \geqq 2)$. Then the tensor product $K_1 \otimes_k \cdots \otimes_k K_n$ over k is a Hilbert ring such that every maximal ideal is of height $t - \max_i\{t_i\}$, where $t = t_1 + \cdots + t_n$.

Proof. Let K'_i is a subfield of K_i such that K'_i is purely transcendental over k. Then $D = K_1 \otimes_k \cdots \otimes_k K_n$ is integral over $C = K'_1 \otimes_k \cdots \otimes_k K'_n$, and C is a universally catenary noetherian domain. Theorem 2 implies that C is a Hilbert ring and every maximal saturated chain of prime ideals in C has same length $t - \max_i\{t_i\}$. Therefore D satisfies the assertion of Corollary 3.

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