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Topological Classification of the Scattered Countable Metric Spaces of Length 3

by

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Abstract

Based upon a general theory we shall present a topological classification of the scattered countable metric spaces of length 3. The number of atoms of length 4 is also given.

1. Preliminaries. Let us start with Cantor's well-known process of deriving. (cf Kuratowski [1]) Let X be a topological space. Let $X^{(0)} = X$ and $X_{(0)}$ the set of the isolated points of $X^{(0)}$. If β is a non-limit ordinal, let $X^{(\beta)} = X^{(\beta-1)} - X_{(\beta-1)}$ and $X_{(\beta)}$ the set of the isolated points of $X^{(\beta)}$, where $\beta - 1$ means the ordinal preceding β. If β is a limit ordinal, let $X^{(\beta)} = \bigcap_{\gamma<\beta} X^{(\gamma)}$ and $X_{(\beta)}$ the set of the isolated points of $X^{(\beta)}$.

Each $X^{(\beta)}$ is a closed subset of X, and each $X_{(\beta)}$ is a discrete open subset of $X^{(\beta)}$.

A space X is called *scattered* if $X^{(\alpha)} = \emptyset$ for some α. The first ordinal α for which $X^{(\alpha)}$ vanishes is called the *length* of the scattered space X and is denoted by leng(X).

The following properties of a scattered space X will be used in this paper implicitly and frequently. Let β be an ordinal and U an open set of X.

(1) $X^{(\beta)} \cap U = U^{(\beta)}$ and $X_{(\beta)} \cap U = U_{(\beta)}$ (, and hence we have the following two).

(2) leng(U) $= \beta$ if and only if $U \cap X^{(\beta)} = \emptyset$ and $U \cap X^{(\gamma)} \neq \emptyset$ for every $\gamma < \beta$.

(3) $X_{(\beta)}$ is dense in $X^{(\beta)}$.

A scattered countable metric space X of length α has the following properties.

(4) The length α is a countable or finite ordinal. (For compact case, α is in addition a non-limit ordinal)

(5) If $\beta + 1 < \alpha$ then $|X_{(\beta)}| = \omega$ with ω the first countable ordinal identified with the countable cardinal. If $\beta + 1 = \alpha$ then $|X^{(\beta)}| = |X_{(\beta)}| \leq \omega$. (For compact case, $|X^{(\beta)}| = |X_{(\beta)}| < \omega$ furthermore.)

If the length $\alpha > 0$ is a non-limit ordinal and $|X^{\alpha-1}| = \beta$, $1 \leq \beta \leq \omega$, the pair (α, β) is called the *type* of X.

As for a compact countable metric space X, the Mazurkiewicz-Sierpiński theorem ([2], also see [1]) says that the topological type of X is uniquely determined by its type (α, n) $1 \leq n < \omega$.

2. General theory.

Definition 1. Let X be a 0-dimensional metric space and p a point of X. X is said to be *self-similar at p* if every clopen set containing p is homeomorphic to X.

Proposition 1. *X is self-similar at p if for any open neighborhood U of p there is a clopen set V of X such that* $p \in V \subseteq U$ *and* $V \approx X$.

Proof. First note that a homeomorphism $f : X \to V$ can be taken so that $f(p) = p$. Indeed if not, say $f(p) = q \neq p$, take disjoint clopen neighborhoods O_p, O_q of p, q respectively so that $f(O_p) = O_q$ and $O_p \cup O_q \subseteq V$, define a homeomorphism $g : V \to V$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in O_p \\ f^{-1}(x) & \text{if } x \in O_q \\ x & \text{if otherwise} \end{cases}$$

and redefine $f' = g \circ f$. Let W be a clopen set of X containing p. To show $W \approx X$ let $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ be a clopen neighborhood base of p. Take m_1 so that $U_{m_1} \subseteq W$ and take a clopen set $V_1 \subseteq U_{m_1}$ containing p and homeomorphic to X, with $h_1 : X \to V_1$ a homeomorphism not moving p. Then take $m_2 > m_1$ so that $U_{m_2} \subseteq V_1 - h_1(X - W)$ and take a clopen set $V_2 \subseteq U_{m_2}$ containing p and homeomorphic to V_1, with $h_2 : V_1 \to V_2$ a homeomorphism not moving p. Further take $m_3 > m_2$ so that $U_{m_3} \subseteq V_2 - h_2 \circ h_1(X - W)$ and take a clopen set $V_3 \subseteq U_{m_3}$ containing p and homeomorphic to V_2, with $h_3 : V_2 \to V_3$ a homeomorphism not moving p.

Repeating this process we have a sequence $m_1 < m_2 < m_3 < \cdots$ and a homeomorphism $h_k \circ h_{k-1} \circ \cdots \circ h_1 : X - W \to h_k \circ h_{k-1} \circ \cdots \circ h_1(X - W) \subseteq U_{m_k} - U_{m_{k+1}}$ for each k. We can now define a homeomorphism $h : X \to W$ by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in X - W \\ h_k(x) & \text{if } x \in h_{k-1} \circ h_{k-2} \circ \cdots \circ h_1(X - W) \\ x & \text{if otherwise.} \end{cases}$$

Thus $X \approx W$, which completes the proof.

Definition 2. Let $\alpha > 0$ be a non-limit ordinal and let X be a scattered countable metric space of type (α, 1) with $\{p\} = X^{(\alpha-1)}$. X is called an atom of length α if X is self-similar at p. A topological sum of at most countably many homeomorphic atoms is called a *molecule*. *A* molecule of the form

$$\overbrace{A \oplus A \oplus \cdots \oplus A}^{n}$$

with A an atom and $1 \leq n < \omega$ is denoted by nA . A molecule of the form

$$\overbrace{A \oplus A \oplus A \oplus \cdots}^{\omega}$$

with A an atom is denoted by ωA. A molecule M homeomorphic to βA with A an atom and $1 \leq \beta \leq \omega$ is called an *A-molecule* . The β is called the *width* of M and denoted by wid(M) .

Examples. The atom of length 1 is the one point space. To count the atoms of length 2 , let X be a scattered countable metric space of type (2, 1) with $X^{(1)} = X_{(1)} = \{p\}$. Then X admits just three topological types. Each type is characterized by the existence of a clopen neighborhood base $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ of p satisfying

(r) $|U_m - U_{m+1}| = 1$ for every m, or

(r') $|U_1 - U_2| = \omega$ and $|U_m - U_{m+1}| = 1$ for every $m \geq 2$, or

(s) $|U_m - U_{m+1}| = \omega$ for every m .

Type r, type r' and type s correspond to compact case, non-compact locally compact case and non-locally compact case, respectively. The X's which admit clopen neighborhood bases satisfying (r), (r'), (s) are respectively denoted by r , r' , s . Consequently the atoms of length 2 are r and s .

Definition 3. A space X is said to *absorb* a space Y if $X \approx X \oplus Y$. In particular, if X is an atom of length α with $\{p\} = X^{(\alpha-1)}$, X absorbs Y if and only if X includes a clopen set not containing p and homeomorphic to Y . Thus, if a molecule X includes a clopen set homeomorphic to a molecule Y with leng(Y) < leng(X) , then X absorbs Y .

If $3 \leq \alpha < \omega_1$ there are infinitely many scattered countable metric spaces of type (α, 1) . However we have

Theorem 1. *Let $\alpha > 0$ be a finite ordinal. Then the number of atoms having length α is finite.*

Theorem 2. *Let $\alpha > 0$ be a finite ordinal and let X be a scattered countable metric space of length α . Then every point p of X has a clopen neighborhood which is self-similar at p .*

Theorem 3. *Let $\alpha > 0$ be a finite ordinal and let X be a scattered countable metric space of length α . Then X has a decomposition D consisting of finitely many clopen molecules such that*

(*) *for each atom A , at most one A-molecule is a member of D , and each member of D does not absorb the other member of D .*

The decomposition D is unique in the sense that if D' is another such decomposition then there is a bijection $\Phi : D \to D'$ satisfying $M \approx \Phi(M)$ for every $M \in D$.

Examples. Theorem 1 and 2 do not hold if the length $\alpha > \omega$. Put $X = [0, \omega^{\omega}]$ and $A_n = X - X_{(n)}$, $n = 1, 2, 3, \ldots$, the subspace of X obtained by removing the limit ordinals whose cofinality is ω^n . Then each A_n is an atom of length $\omega + 1$, and if $n < m$ then $A_n \not\approx A_m$ because

$$(A_n)_{(n-1)} \cup (A_n)_{(n)} = X_{(n-1)} \cup X_{(n+1)} \approx \omega s$$
$$\text{but} \quad (A_m)_{(n-1)} \cup (A_m)_{(n)} = X_{(n-1)} \cup X_{(n)} \approx \omega r .$$

As for Theorem 2, using A_n above, define $B_n = A_n - \{\omega^{\omega}\}$ and $Y = (\oplus_{n=1}^{\infty} B_n) \cup \{p\}$ with the topology such that the topology of $\oplus_{n=1}^{\infty} B_n$ is not disturbed and $U_m = (\oplus_{n=m}^{\infty} B_n) \cup \{p\}$, $m = 1, 2, 3, \ldots$, is a clopen neighborhood base of the new point p . Then Y is a scattered countable metric space of type $(\omega + 1, 1)$ with $\{p\} = Y^{(\omega)}$. The point p has no clopen neighborhood in Y which is self-similar at p . Indeed, $U_m \not\approx U_{m+1}$ for every m because

$$(U_m)_{(m-1)} \cup (U_m)_{(m)} \approx \omega s \oplus \omega r \quad \text{but} \quad (U_{m+1})_{(m-1)} \cup (U_{m+1})_{(m)} \approx \omega r .$$

To make the proof go smooth we shall give two easy technical lemmas.

Lemma 1. *Let X, R be spaces and p a point of X . Let $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ be a clopen neighborhood base of p . Assume each $U_m - U_{m+1}$ is written*

$$U_m - U_{m+1} = X_m^0 \cup X_m^1 \cup \cdots \cup X_m^{k_m} \ (k_m = 0 \text{ may happen})$$

by finitely many mutually disjoint clopen sets X_m^i, $0 \le i \le k_m$, of X such that

$$X_m^1 \approx X_m^2 \approx \cdots \approx X_m^{k_m} \approx R .$$

If $|\{m|\ k_m \ge 1\}| = \omega$ then there is a clopen neighborhood base $X = V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ of p satisfying

$$V_m - V_{m+1} = X_m^0 \cup R_m$$

for every m , where R_m is a clopen set of X such that $X_m^0 \cap R_m = \emptyset$ and $R_m \approx R$.

Proof. Rewite $\{X_m^i \mid m = 1, 2, 3, \ldots, 1 \le i \le k_m\} = \{X_1, X_2, X_3, \ldots\}$ so that if $X_m^i = X_n$, $X_{m'}^{i'} = X_{n'}$ *and* $m < m'$ *then* $n < n'$. We have only to put

$$V_m = \{p\} \cup (\cup_{j=m}^{\infty} X_j^0) \cup (\cup_{j=m}^{\infty} X_j) .$$

Notation. We use the notation $M \overset{h}{\in} D$ to mean that D contains a member homeomorphic to M .

Lemma 2. *Let X be a scattered countable metric space of finite length and let D* *be a decomposition of X into finitely many clopen molecules satisfying* (*) *of Theorem* 3. *Let M be a clopen A-molecule of X (not necessarily satisfying* $M \overset{h}{\in} D$*) with A an atom. Then M is absorbed by a member N of D with leng*(*N*) > leng(*M*) *or D contains, as a member, an A-molecule of width not smaller than the width of M* .

Proof. Let $\alpha = \text{leng}(M)$. If $\text{wid}(M) = \omega$ (which is equivalent to $|M^{(\alpha-1)}| = \omega$), writing $M^{(\alpha-1)} = \{x_1, x_2, x_3, \ldots\}$, decompose M as $M = \bigcup_{i=1}^{\infty} A_i$ with A_i a clopen atom homeomorphic to A and satisfying $\{x_i\} = A_i^{(\alpha-1)}$. Since $|D| < \omega$, some member N of D contains countably many elements, say $x_{i_1}, x_{i_2}, x_{i_3}, \ldots$, of $M^{(\alpha-1)}$. Put $M' = \bigcup_{j=1}^{\infty}(A_{i_j} \cap N)$. Then M' is a clopen molecule homeomorphic to M and included in N . If leng$M$ = leng$N$ then $M \approx N$. If leng$M <$ leng N then M is absorbed by N by the remark following Definition 3.

If wid(M) is finite, also writing $M^{(\alpha-1)} = \{x_1, x_2, \ldots, x_k\}$, decompose M as $M = \bigcup_{i=1}^{k} A_i$ with A_i a clopen atom homeomorphic to A and satisfying $\{x_i\} = A_i^{(\alpha-1)}$. Take $N_i \in D$ so that $x_i \in N_i$, then N_i includes a clopen atom $A_i \cap N_i$ homeomorphic to A . If leng(M) < leng(N_i) for some i , then N_i absorbs A and hence M because wid$M < \omega$. If leng(M) = leng(N_i) for every i , then N_i should be an A-molecule for every i . Since an A-molecule appeas at most once as a member of D , we have $N_1 = N_2 = \cdots = N_k$ so that wid(M) $\leq$ wid(N_1). This completes the proof.

Proof of Theorem 1, 2 and 3. We shall prove Theorem 1, 2 and 3 simultaneously by induction on α. These therems are trivially true if $\alpha = 1$. Let γ be a finite ordinal and assume Theorem 1, 2 and 3 are valid for for every $\alpha < \gamma$. To first show Theorem 2 for γ , let X be a scattered countable metric space of length γ and p a point of X . Let $p \in X_{(\beta)}$ and, using 0-dimensionality, take a clopen set U of X so that $U \cap X^{(\beta)} = \{p\}$.
If $\beta < \gamma - 1$ then leng(U) $\leq \gamma - 1$ so that induction hypothesis assures the existence of a clopen neighborhood V of p included in U and self-similar at p . Thus we may assume that type $X = (\gamma, 1)$ and $\{p\} = X^{(\gamma-1)} = X_{(\gamma-1)}$. Let $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ be a clopen neighborhood base of p . Since leng($U_m - U_{m+1}$) $< \gamma$ it follows from induction hypothesis that each $U_m - U_{m+1}$ has a decomposition D_m consisting of finitely many clopen molecules and satisfying (*) . Clearly each member of D_m is of length less than γ . Now define a equivalence relation ~ on the set {1, 2, 3, . . . } as follows : $m \sim m'$ if and only if for each atom A , $\omega A \overset{h}{\in} D_m$ is equivalent to $\omega A \overset{h}{\in} D_{m'}$, and $nA \overset{h}{\in} D_m$, $1 \leq n < \omega$, is equivalent to $n'A \overset{h}{\in} D_{m'}$, $1 \leq n' < \omega$. ($n \neq n'$ may happen.)

Note that the number of equivalence classes by $\sim$ is finite because the number of atoms of length less than γ is finte by induction hypothesis. We can thus take l so that

$$|C \cap \{l, l+1, l+2, \ldots\}| = 0 \text{ or } \omega$$

for every equivalence class C .

We shall prove that U_l is self-similar at p . For convenience let U_l be renamed X , let

U_{l+m-1} be renamed U_m , $m = 1, 2, 3, \ldots$, and let D_{l+m-1} be renamed D_m , $m = 1, 2, 3, \ldots$. Let

$$A_1, A_2, \ldots, A_k$$

be all the atoms of length less than γ so arranged that if $i \leq j$ then leng(A_i) $\geq$ leng(A_j) . Recalling how we took l we see that for each $1 \leq i \leq k$ one and only one of the following three cases occurs :

(a_i) $\omega A_i \overset{h}{\in} D_m$ for countalbly many m' s .

(b_i) $\omega A_i \overset{h}{\notin} D_m$ for every m , and $nA_i \overset{h}{\in} D_m$, $1 \leq n < \omega$, for countably many m' s (with n maybe varying).

(c_i) $\omega A_i, nA_i \overset{h}{\notin} D_m$ for every m and $1 \leq n < \omega$.

Using Lemma 1 we shall remake U_m and D_m (at most) k times as follows : First consider the case $i = 1$. If (c_1) occurs there is nothing to do. If (a_1) does, apply Lemma 1 with $R = \omega A_1$ and $k_m = 0$ or 1 to remake U_m , $m = 1, 2, 3, \ldots$, so that $U_m - U_{m+1}$ has a decomposition $\tilde{D}_m$ satisfying :

(d) $\tilde{D}_m$ contains only one member homeomorphic to ωA_1 and no member homeomorphic to nA_1 , $1 \leq n < \omega$.

(e) The members of D_m coincide with those of $\tilde{D}_m$ except for A_1-molecules.

If (b_1) occurs, apply Lemma 1 with $R = A_1$ to remake U_m , $n = 1, 2, 3, \ldots$, so that $U_m - U_{m+1}$ has a decomposition $\tilde{D}_m$ satisfying :

(d') $\tilde{D}_m$ contains only one member homeomorphic to A_1 and no member homeomorphic to βA_1 , $2 \leq \beta \leq \omega$.

(e') The members of D_m coincide with those of $\tilde{D}_m$ except for A_1-molecules.

In either case, $\tilde{D}_m$ may not satisfy the latter half of the condition (*) . To avoid unnecessary discussion, do not make a new decomposition of $U_m - U_{m+1}$ so that (*) is satisfied. Let $\tilde{D}_m$ be renamed D_m again.

Repeat this modification (at most) k times until ending at A_k , where A_k is, of course, the one point space. Then the U_m , D_m thus obtained satisfy the following :

(f) D_m coincides with $D_{m'}$ for every m, m' in the sense that there is a bijection $\Phi : D_m \to D_{m'}$ satisfying $M \approx \Phi(M)$ for every $M \in D_m$. In particular $U_m - U_{m+1} \approx U_{m'} - U_{m'+1}$ for every m, m' .

(g) For each $1 \leq i \leq k$, D_m contains at most one A_i-molecule, and this A_i-molecule is homeomorphic to ωA_i or A_i . (This is not necessary here but will be used later.)

It follows from (f) and Proposition 1 that U_l (renamed X) is self-similar at p , which completes the proof of Theorem 2 for γ .

To use later let D_m be finally modified so that (f), (g) and (*) hold simultaneously. Let $M_1, M_2, \ldots, M^t$ be all the members of D_m arranged so that if $i \leq j$ then leng(M^i) $\geq$ leng(M^j) . Define D^i_m , $0 \leq i \leq t$, inductively as follows : $D^0_m = D_m$ and

$$D_m^i = \begin{cases} D_m^{i-1} & \text{if } M^i \notin D_m^{i-1} \\ \{M^i \cup (\cup E^i)\} \cup (D_m^{i-1} - E^i) & \text{if } M^i \in D_m^{i-1} \end{cases}$$

where E^i denotes the members of D_m^{i-1} absorbed by M^i and $\cup E^i$ denotes the union of members of E^i . Of course $M^i \cup (\cup E^i) \approx M^i$ .Let D_m^t be renamed D_m again. It is easy to see that the new D_m , $m = 1, 2, 3, \ldots$, satisfy (f), (g) and (*) .

To show Theorem 1 for γ , let

$$A_1, A_2, \ldots, A_k$$

be all the atoms of length less than γ as above, and let ρ_γ be the number of atoms of length γ . We shall prove

$$(\#) \quad \rho_\gamma \leq 3^k .$$

Note that we have already proved above that a scattered countable metric space X of type $(\gamma, 1)$ with $\{p\} = X^{(\gamma-1)}$ includes a clopen set U containing p and admitting a clopen neighborhood base $U = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ of p for which each $U_m - U_{m+1}$ has a finite decompsition D_m into clopen molecules satisfying (f), (g) and (*) . If X is an atom of length γ we can identify X with U . Since $U_1 - U_2 \approx U_m - U_{m+1}$ for every m , the number of topological types of U (and hence of X) is not greater than that of $U_1 - U_2$. By (g) , the number of topological types of $U_1 - U_2$ is not greater than 3^k , where for each $1 \leq i \leq k$, the ' 3 ' corresponds to the three cases, $\omega A_i \overset{h}{\in} D_1$, $A_i \overset{h}{\in} D_1$ and $\omega A_i, A_i \overset{h}{\notin} D_1$. Consequently (#) follows. This completes the proof of Theorem 1 for γ .

Remark. The inequality (#) is far from a good estimate because the right side counts many impossible combinations of molecules with respect to the condition (*) .

To finally show Theorem 3 for γ , let X be a scattered countable metric space of length γ . Using 0-dimensionality we have a discrete family $\{A_x \mid x \in X^{(\gamma-1)}\}$ of clopen sets of X satisfying $x \in A_x$ for each $x \in X^{(\gamma-1)}$. By Theorem 2 we can assume A_x is an atom of length γ . Gathering homeomorphic atoms, we obtain finitely many mutually disjoint clopen molecules $M_1, M_2, \ldots, M_m$ of length γ such that $M_1 \cup M_2 \cup \cdots \cup M_m = \cup_{x \in X^{(\gamma-1)}} U_x$, and for each atom A of length γ , an A-molecule appears at most once in $M_1, M_2, \ldots, M_m$. By induction hypothesis $X - (M_1 \cup M_2 \cup \cdots \cup M_m)$ has a decomposition D consisting of finitely many clopen molecules of X satisfying (*) . For each $1 \leq i \leq m$ define E_i inductively as follows :

$$\begin{cases} E_1 \text{ is the members of } D \text{ absorbed by } M_1 ; \\ E_i \text{ is the members of } D - (\cup_{j=1}^{i-1} E_j) \text{ absorbed by } M_i . \end{cases}$$

Define

$$D' = \{M_i \cup (\cup E_i) \mid 1 \leq i \leq m\} \cup (D - \cup_{i=1}^{m} E_i)$$

with $\cup E_i$ denoting the union of members of E_i . Then D' is a desired decomposition of X satisfying (*) .

To check the uniequness let D, D' be two such decompositions of X and suppose to the contrary that there is $\beta \leq \gamma$ admitting an atom A of length β and an A-molecule M such that

$$M \overset{h}{\in} D \text{ and } M \overset{h}{\notin} D' \text{, or } M \overset{h}{\notin} D \text{ and } M \overset{h}{\in} D' .$$

We may assume the β is the largest one satisfying these conditions. If, for instance, the former happens then $M \overset{h}{\in} D$ and (*) say that M is not absorbed by any member of D and hence of D' whose length greater than β . Thus by Lemma 2, D' contains an A-molecule N satisfying wid $M \leq$ wid N so that wid M <wid N because the equality would imply $M \approx N$. Then $N \in D$ and (*) say again that N is not absorbed by any member of D of length greater than β . Thus by Lemma 2 again, D contains an A-molecule L with wid $N \leq$ wid$L$ so that wid M <wid L . This implies that two different A-molecules M, L are members of D, which contradicts (*). This completes the proof of Theorem 3 for γ . We have thus finished the proof of Theorems 1, 2 and 3.

The following corollary is a key to counting the number of atoms of length 3 and 4 .

Corollary 1. *Let* $2 \leq \alpha < \omega$ *and let* X *be a scattered countable metric space of type* $(\alpha, 1)$ *with* $\{p\} = X^{(\alpha-1)}$. *Then* X *is an atom if and only if* p *has a clopen neighborhood base* $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ *satisfying* :

(1) $U_m - U_{m+1} \approx U_{m'} - U_{m'+1}$ *for every* m, m' .

(2) *If we decompose* $U_1 - U_2$ *into finitely many clopen molecules satisfying* (*) *of Theorem* 3, *then every member* M *of the decomposition is of the form* $M \approx \omega A$ *or* $M \approx A$ *with* A *an atom*.

The topological type of $U_1 - U_2$ *is uniquily determined.*

Remark. Condition (2) is indispensable for the uniqueness as the following trivial example shows : Let $X = [0, \omega]$ and $U_m = [m, \omega]$, $U'_m = [2m, \omega]$, $m = 0, 1, 2, \ldots$.

Proof of Corollary 1. The ‘if ’ part is assured by Proposition 1 . The existence of such U_m has already been verified in the proof of Theorem 2 above. To show the uniqueness let U'_m , $m = 1, 2, 3, \ldots$, be another such neighborhood base of p and let D, D' be the decompositions of $U_1 - U_2$ and $U'_1 - U'_2$ respectively satisfying (*) . We first prove the assertion that if $M \in D$ then $U'_1 - U'_2$ includes a clopen set homeomorphic to M , and if $M' \in D'$ then $U_1 - U_2$ includes a clopen set homeomorphic to M' . In case M is a single atom of length β , let $\{a\} = M^{(\beta-1)}$, take m so that $a \in U'_m - U'_{m+1}$ and take a clopen neighborhood U of a included in $M \cap (U'_m - U'_{m+1})$. Then $U \approx M$ because M is an atom.

It follows from (1) that $U'_1 - U'_2$ includes a clopen set homeomorphic to U and hence to M . If M is not a single atom then, by (2), M is of the form $M = \bigcup_{i=1}^{\infty} A_i$ with A_i

mutually disjoint clopen atoms homeomorphic to a commom atom A of length β . Write $M^{(\beta-1)} = \{x_1, x_2, x_3, \ldots\}$ with $x_i \in A_i$ for each i . Take m so that $M \cap U'_m = \emptyset$, take $k < m$ so that $|M^{(\beta-1)} \cap (U'_k - U'_{k+1})| = \omega$ and, writing $M^{(\beta-1)} \cap (U'_k - U'_{k+1}) = \{x_{i1}, x_{i2}, x_{i3}, \ldots\}$, put $U = (\cup_{j=1}^{\infty} A_{ij}) \cap (U'_k - U'_{k+1})$. Then $M \approx U \subseteq U'_k - U'_{k+1}$. It follows from (1) that $U'_1 - U'_2$ includes a clopen set homeomorphic to U and hence to M .

Quite similarly we can find a clopen set of $U_1 - U_2$ homeomorphic to M' . This completesthe proof of the assertion. Now suppose to the contrary that $U_1 - U_2 \not\approx U'_1 - U'_2$ so that there is $\beta \leq \alpha - 1$ admitting an atom A of length β and an A-molecule M such that

$$M \overset{h}{\in} D \text{ and } M \overset{h}{\notin} D' \text{, or } M \overset{h}{\notin} D \text{ and } M \overset{h}{\in} D' .$$

Combined with the assertion above, this however leads to a contradiction in the same way as in the last part of the proof of Theorem 3. This completes the proof of Corollary 1.

The first easy application of Theorem 3 is the following.

Proposition 2. *Let X be a scattered countable metric space of type* $(2, n)$, $1 \leq n < \omega$ *. Then X admits just* $n + 2$ *topological types as follows* :

$$nr,\ ns,\ kr \oplus (n-k)s,\ 1 \leq k \leq n-1,\ \text{and } nr \oplus \mathbb{N}$$

with $\mathbb{N}$ *the countable discrete space.*

Note that a finite points space is absorbed by nr and ns , and that $\mathbb{N}$ is absorbed by ns but not by nr .

Proposition 3. *Let X be a scattered countable metric space of type* $(2, \omega)$ *. Then X is homeomorphic to one and only one of the following spaces* :

$$\omega r,\ \omega s,\ kr \oplus \omega s,\ 1 \leq k < \omega,\ ks \oplus \omega r,\ 1 \leq k < \omega,\ \text{and } \omega r \oplus \omega s .$$

Note that $\mathbb{N}$ is absorbed by ωr as well as by ωs so that $\mathbb{N}$ does not appear in the decomposition.

3. Classification. Let us start with counting the number of atoms of length 3 .

Theorem 4. *The number of atoms of length* 3 *is nine*.

Proof. Let X be an atom of length 3 with $\{p\} = X^{(2)} = X_{(2)}$ and let $X = U_1 \supsetneq U_2 \supsetneq U_3 \supsetneq \cdots$ be a clopen neighborhood base of p satisfining (1), (2) in Corollary 1. By virtue of the uniqueness of $U_1 - U_2$ we have only to count the topological types of $U_1 - U_2$. Let D be the finite decomposition of $U_1 - U_2$ into clopen molecules satisfying (*) . By (2) of Corollary 1, the molecules which may appear as members of D are the following six :

$$r,\ \omega r,\ s,\ \omega s,\ \text{the one point space and } \mathbb{N} .$$

Searching the possible combinations of the six molecules satisfying (*), we obtain the following nine topological types of $U_1 - U_2$.

atoms of length 3	examples in $[0, \omega_1)$	residues	note
$X(r)$	$[0, \omega^2]$	$\emptyset$	c, rh
		$\mathbb{N}$	lc, rh
		ns , $1 \leq n < \omega$	
		ωr	lc, rh
		ωs	
		$ns \oplus \omega r$, $1 \leq n < \omega$	
		$\omega r \oplus \omega s$	
$X(r')$	$[0, \omega^2] - \{\omega(2n-1) \mid 1 \leq n < \omega\}$	$\emptyset$	rh
		ns , $1 \leq n < \omega$	
		ωr	rh
		ωs	
		$ns \oplus \omega r$, $1 \leq n < \omega$	
		$\omega r \oplus \omega s$	
$X(s)$	$[0, \omega^3] - C_\omega$	$\emptyset$	rh
		nr , $1 \leq n < \omega$	
		ωr	
		ωs	rh
		$ns \oplus \omega s$, $1 \leq n < \omega$	
		$\omega r \oplus \omega s$	
$X(\mathrm{r} \oplus s)$	$[0, \omega^3] - C_\omega) \cup$ $\{\omega^2 n + \omega \mid n < \omega\}$	$\emptyset$	
		ωr	
		ωs	
		$\omega r \oplus \omega s$	
$X(\omega r)$	$[0, \omega^3] - C_{\omega^2}$	$\emptyset$	rh
		ns , $1 \leq n < \omega$	
		ωs	
$X(\omega s)$	$[0, \omega^4] - (C_\omega \cup C_{\omega^3})$	$\emptyset$	rh
		nr , $1 \leq n < \omega$	
		ωr	
$X(\mathrm{s} \oplus \omega r)$	$[0, \omega^3] - \{\omega^2(2n-1) + \omega m \mid$ $1 \leq n < \omega, m < \omega\}$	$\emptyset$	
		ωs	
$X(r \oplus \omega s)$	$([0, \omega^4] - (C_\omega \cup C_{\omega_3})) \cup$ $\{\omega^3 n + \omega \mid n < \omega\}$	$\emptyset$	
		ωr	
$X(\omega r \oplus \omega s)$	$([0, \omega^4] - (C_\omega \cup C_{\omega_3})) \cup$ $\{\omega^3 m + \omega^2 n + \omega \mid m < \omega, \mathrm{n} < \omega\}$	$\emptyset$	

Table 1

$$r\ ,\ r \oplus \mathbb{N}\ ,\ s\ ,\ r \oplus s\ ,$$
$$\omega r\ ,\ \omega s\ ,\ s \oplus \omega r\ ,\ r \oplus \omega s\ ,\ \omega r \oplus \omega s.$$

Consequently the molecule N appears only in $r \oplus \mathbb{N} = r'$ because, in the other seven cases, $\mathbb{N}$ is always absorbed. This completes the proof of Theorem 4 .

Let

$$X(r)\ ,\ X(r')\ ,\ X(s)\ ,\ X(r \oplus s)$$
$$X(\omega r)\ ,\ X(\omega s)\ ,\ X(s \oplus \omega r)\ ,\ X(r \oplus \omega s)\ ,\ X(\omega r \oplus \omega s)$$

denote the corresponding topological types of X .

Let X be a scattered coutable metric space of type (3, 1) and D the finite decomposition of X into clopen molecules satisfying (*) . By virtue of the uniqueness of D , to count the topological types of X is to count the decompositions D . The decomposition D is of the form

$$D = \{A\} \text{ or } D = \{A\} \cup \{M_\lambda \mid \lambda \in \Lambda\}\ ,$$

where A is homeomorphic to one of the nine atoms above and M_λ is a molecule of length less than 3 . Let us call $X - A$ the *residue* of A . Each M_λ is homeomorphic to one of the following :

nr , $1 \leq n < \omega$, ωr , ns , $1 \leq n < \omega$, ωs , $\mathbb{N}$ and the finite points spaces .

Choosing the possible combinations among them so that (*) is satisfied, we have Table 1 giving topological classification of the scattered countable metric spaces of type (3, 1) . (Recall that, as stated after Definition 3, an atom A of length 3 with $\{p\} = A^{(2)}$ absorbs an molecule M of length less than 3 if and only if A includes a clopen set homeomorphic to M and not containing p .) In the table, C_β, $\beta = \omega, \omega^2, \omega^3$, denotes the set of ordinals less than ω_1 whose cofinality is β . The topology of each example in $[0, \omega_1)$ is that induced from the order topology on $[0, \omega_1)$. The symbols c, lc, rh mean respectivelly compact, locally compact, rankwise homogeneous. A scattered space X is defined to be *rankwise homogeneous* if for each ordinal β and $x, x' \in X(\beta)$ there is a homeomorphism h : $X \to X$ sending x to x' .

Let us go on to the type $(3, k)$, $1 \leq k < \omega$. Let A_i , $1 \leq i \leq 9$, denote in order the nine atoms of length 3 and for each i , R_i the set of residues of A_i listed in the table above. For example, $A_1 = X(r)$ and

$$R_1 = \{\emptyset, \mathbb{N}, \omega r, \omega s, \omega r \oplus \omega s\} \cup \{ns \mid 1 \leq n < \omega\} \cup \{ns \oplus \omega r \mid 1 \leq n < \omega\}\ .$$

Theorem 5. *Let X be a scattered countable metric space of type* $(3, k)$, $1 \leq k < \omega$. *Then X can be written uniquely as*

$$X \approx A_{i1} \oplus A_{i2} \oplus \cdots \oplus A_{ik} \oplus R\ ,$$

where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq 9$ *and* $R \in R_{i1} \cap R_{i2} \cap \cdots \cap R_{ik}$.

In the case of type $(3, \omega)$, almost molecules of length less than 3 are absorbed and vanish.

Theorem 6. *Let X be a scattered countable metric space of type* $(3, \omega)$ *. Then X can*
be written uniquely as

$$X \approx \oplus_{i=1}^{\infty} X_j \oplus R ,$$

where $X_j \in \{A_1, A_2, \dots, A_9\}$ *and*

$$R \in \{\emptyset, \omega r, \omega s\} \cup \{nr \mid n < \omega\} \cup \{ns \mid n < \omega\} ;$$

$R = nr$ is possible only when $X_j = A_3$ or A_6 for every j ,
$R = ns$ is possible only when $X_j = A_1$ or A_2 or A_5 for every j ,
$R = \omega r$ is possible only when $X_j = A_1$ or A_2 or A_4 or A_8 for finitely many j 's and $X_j = A_3$ or A_6 for the other j 's and
$R = \omega s$ is possible only when $X_j = A_3$ or A_4 or A_7 for finitely many j 's and $X_j = A_1$ or A_2 or A_5 for the other j 's .

4. The number of atoms of length 4 . Let ρ_n denote the number of atoms of length n . As verified before, $\rho_1 = 1$, $\rho_2 = 2$, $\rho_3 = 9$. Compared with ρ_3 , the number ρ_4 is considerably large. In fact a rough calculation gives at least

$$\rho_4 \geq 3^9 - 1 = 19682 .$$

This inequality is obtained in the following way. Let X be an atom of length 4 with $\{p\} = X^{(3)}$ and let $X = U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ be a clopen neighborhood base of p satisfying (1), (2) of Corollary 1. Assume further that the finite decomposition D of $U_1 - U_2$ into clopen molecules satisfying (*) **contains no molecule of length less than 3 .** Then the number of topological types of $U_1 - U_2$ is $3^9 - 1$, the right side of the inequality, where foreach $1 \leq i \leq 9$, the '3' coresponds to the three cases, $A_i \overset{h}{\in} D$, $\omega A_i \overset{h}{\in} D$ and $A_i, \omega A_i \overset{h}{\notin} D$.

To determine ρ_4 , we should take account of the molecules of length less than 3 which may appear in the decomposition. Consider the following table.

molecules	nonabsorbers
$\mathbb{N}$	A_1
r	$A_3, \omega A_3, A_6, \omega A_6$
s	$A_1, \omega A_1, A_2, \omega A_2, A_5, \omega A_5$
ωr	$A_1, A_2, A_3, \omega A_3, A_4, A_6, \omega A_6, A_8$
ωs	$A_1, \omega A_1, A_2, \omega A_2, A_3, A_4, A_5, \omega A_5 , A_7$

Table 2

The molecules in the first column are those of length less than 3 which can appear as memebers of the decomposition of $U_1 - U_2$. The second row, for example, means A_3, ωA_3 , A_6, ωA_6 do not absorb r but the others do.

Table 2 tells us :

(1) The following pairs of molecules can not appear simaltaneously as members of the decomposition of $U_1 - U_2$.

$$\mathbb{N} \,\&\, r \,,\ \mathbb{N} \,\&\, s \,,\ \mathbb{N} \,\&\, \omega r \,,\ \mathbb{N} \,\&\, \omega s \,,\ r \,\&\, s \,,\ r \,\&\, \omega r \,,\ s \,\&\, \omega s$$

Indeed, $\mathbb{N}$ and r have no common nonabsorber ; $\mathbb{N}$ is absorbed by s , ωr and ωs ; r and s have no common nonabsorber ; r and ωr are both r-molecules ; s and ωs are both s-molecules.

(2) r & ωs appear simultaneously only if A_3 appears and the others do not appear.

(3) s & ωr appear simultaneously only if one or two of A_1, A_2 appear and the others do not appear.

(4) ωr & ωs appear simultaneously only if one or two or three or four of A_1, A_2, A_3, A_4 appear and the others do not appear.

(5) More than two molecules do not appear simaltaneously because one of them absorbs another.

Thus the number of the decompositions D of $U_1 - U_2$ satisfying (*) and containing at least one molecule of length less than 3 is

$$1 + (3^2 - 1) + (3^3 - 1) + (3^2 2^4 - 1) + (3^3 2^3 - 1) + 1 + (2^2 - 1) + (2^4 - 1) = 412$$

The first five terms correspond to the cases where only one of the five molecules

$$\mathbb{N} \,,\ r \,,\ s \,,\ \omega r \,,\ \omega s$$

appears. The last three terms correspond to the cases discussed in (2), (3), (4) above. Adding 412 to 19682 we have

Theorem 7. $\rho_4 = 20094$.

References.

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