INVARIANT POLYNOMALS WITH THREE MATRIX ARGUMENTS, EXTENDING THE POLYNOMIALS WITH LOWER NUMBERS OF MATRIX ARGUMENTS

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Abstract

We define a class of invariant polynomials $C_\phi^{\kappa,\lambda,\nu}(X, Y, Z)$ with three matrix arguments, through the theory of polynomial representations of $G\ell(m, R)$. These polynomials, extending the zonal polynomials and the invariant polynomials with two matrix arguments due to A.W. Davis, are useful in multivariate distribution theory. Some properties and relations satisfied by the polynomials $C_\phi^{\kappa,\lambda,\nu}$ are shown. The investigation in this paper must be useful for the further generalization to the polynomials $C_\phi^{\kappa_1,\kappa_2,\ \ ,\kappa_r}$ with any number r matrix arguments, $r \geq 4$.

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1. Introduction

The invariant polynomials with r $m \times m$ symmetric matrix arguments X_i, $i=1, \ldots, r$, $r \geq 1$, having the property of invariance under the simultaneous transformations

$$X_i \rightarrow H'X_i H, \; i=1, \ldots, r, \text{ for } H \in O(m), \tag{1.1}$$

where $O(m)$ is the group of $m \times m$ orthogonal matrices, may be defined through the theory of group representations, i.e., the theory of polynomial representations of $G\ell(m, R)$, the group of $m \times m$ real nonsingular matrices. The invariant polynomial with one matrix argument, the zonal polynomial, $C_\kappa(X)$, indexed by the ordered partition κ of the nonnegative integer k into not more than m parts, generates the unique one-dimensional subspace contained in $V_\kappa[X]$ occurring in the decomposition $P_k[X] = \oplus_\kappa V_\kappa[X]$ into irreducible invariant subspaces of $P_k[X]$, the class of homogeneous polynomials of degree k in the elements of X. These polynomials have been discussed and utilized in the derivation of power series expansions of multivariate distributions in normal theory in an extensive literature (see e.g., Constantine [3], Herz [8], James [10] and many others in the recent literature).

Davis [5], [6] extended the zonal polynomials and defined the invariant polynomials $C_\phi^{\kappa,\lambda}(X, Y)$ with two matrix arguments. These satisfy the basic relationship

$$\int_{O(m)} C_\kappa(AH'XH) C_\lambda(BH'YH) dH$$
$$= \sum_{\phi \in \kappa \cdot \lambda} C_\phi^{\kappa,\lambda}(A, B) C_\phi^{\kappa,\lambda}(X, Y) / C_\phi(I), \tag{1.2}$$

where κ, λ and ϕ denote ordered partitions of k, ℓ and $f = k + \ell$ respectively into not more than m parts, and $\phi \in \kappa \cdot \lambda$ signifies that the irreducible representation of $G\ell(m, R)$ indexed by 2ϕ occurs in the decomposition of the Kronecker product $2\kappa\otimes2\lambda$ of the irreducible representations indexed by 2κ and 2λ. $C_{\phi}^{\kappa,\lambda}(X, Y)$ generates the one-dimensional subspace contained in $V_{\Phi,\Phi=2\phi}^{\kappa,\lambda}[X, Y]$ occurring in the decomposition $P_{k,\ell}[X, Y]=P_k[X]\otimes P_\ell[Y]=\oplus_\kappa\oplus_\lambda\oplus_\Phi V_{\phi}^{\kappa,\lambda}[X, Y]$ into irreducible subspaces of $P_{k,\ell}[X, Y]$, the class of homogeneous polynomials of degrees k and ℓ in the elements of X and Y respectively. Some difficulty in distributional problems which could not be solved in terms of zonal polynomials have been solved by these polynomials $C_{\phi}^{\kappa,\lambda}$. Applications of the polynomials in multivariate distribution theory are found in Davis [5], [6] and Chikuse [2].

However, there are still problems unsolved by the $C_{\phi}^{\kappa,\lambda}$; these are (i) the doubly noncentral F distributions with unequal covariance matrices, (ii) the distributions of S_1+S_2 and $S_1+S_2+S_3$, where the S_i are independently distributed as $W_m(n_i, \Sigma_i, \Omega_i)$, i=1, 2, 3, and we assume $\Omega_2=\Omega_3=0$ when we consider $S_1+S_2+S_3$, and (iii) the distributions of the roots of $W=(S_1+S_2)^{-1/2}S_0(S_1+S_2)^{-1/2}$, where the S_i are independently distributed as $W_m(n_i, \Sigma_i, \Omega_i)$, i=0, 1, 2, and we assume $\Omega_i=0$, i=1, 2. The problems (ii) and (iii) arise in the multivariate Behrens-Fisher discriminant analysis and these distributions under more restricted assumptions were derived in terms of the $C_{\phi}^{\kappa,\lambda}$ in Chikuse [2]. In Section 2 we shall define a class of invariant polynomials $C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z)$ with three matrix arguments, extending the $C_{\phi}^{\kappa,\lambda}(X, Y)$, and some elementary properties and fundamental relations satisfied by the $C_{\phi}^{\kappa,\lambda,\nu}$ are shown in Section 3. Section 4 presents Laplace transforms and beta-type integrals, and expansions for incomplete gamma and beta functions. Generalized Laguerre polynomials with three matrix arguments are constructed in Section 5. Some expansions are given along the lines of Davis [5] in Section 6, and finally Section 7 presents some differential identities satisfied by the $C_{\phi}^{\kappa,\lambda,\nu}$.

The definition and the most of the properties and relations of the $C_{\phi}^{\kappa,\lambda,\nu}$ are essentially direct extensions of those of Davis' polynomials. Due to restrictions on space, the details on the construction and the applications in the multivariate distributional problems already cited of the $C_{\phi}^{\kappa,\lambda,\nu}$ will be presented in a subsequent paper. We note that some interest in the polynomials has been shown by people working in the field of econometric theory. Finally we notice that our consideration of the invariant polynomials with three argument matrices must be useful for the further generalization to those $C_{\phi}^{\kappa_1,\kappa_2,\cdots,\kappa_r}(X_1, X_2, \ldots, X_r)$ with any number r matrix arguments.

2. Invariant Polynomials With Three Matrix Arguments

The argument in Davis [6, Section 2] can be directly extended for defining a polynomial $\Gamma_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z)$ which is invariant under the simultaneous transformations

$$X \to H'XH,\ Y \to H'YH,\ Z \to H'ZH,\ H \in O(m), \qquad (2.1)$$

where X, Y, Z are m×m complex symmetric matrices and O(m) denotes the group of m×m orthogonal matrices. We summarize our argument in the following

Let $P_k[X]$ and $P_{k,\ell,n}[X, Y, Z]$ be the classes of homogeneous polynomials of degree k in the elements of X and of degrees k, ℓ and n in the elements of X, Y and Z respectively. In connection with the theory of polynomial representations of $G\ell(m, R)$, the group of m×m real nonsingular matrices (Boerner [1, Chapter 5] and James (unpublished lecture notes)), we have the decompositions into irreducible subspaces of these polynomial classes,

$$P_k[X] = \oplus_{\kappa} V_{\kappa}[X],\ P_{\ell}[Y] = \oplus_{\lambda} V_{\lambda}[Y],\ P_n[Z] = \oplus_{\nu} V_{\nu}[Z],$$

and

$$P_{k,\ell,n}[X,\ Y,\ Z]=P_k\,[X]\otimes P_\ell\,[Y]\otimes P_n\,[Z]$$
$$=\oplus_\varkappa \oplus_\lambda \oplus_\nu \oplus_\Phi V_\Phi^{\varkappa,\lambda,\nu}[X,\ Y,\ Z]. \qquad (2.2)$$

Here, $\varkappa$, λ and ν are ordered partitions of k, ℓ and n respectively into not more than m parts, and Φ runs over all ordered partitions of 2f, $f=k+\ell+n$, indexing the irreducible representations of $G\ell(m,\ R)$ occurring in the decomposition of the Kronecker product $2\varkappa\otimes 2\lambda\otimes 2\nu$ of the irreducible representations indexed by $2\varkappa$, 2λ and 2ν. $V_\varkappa\,[X]$, for example, and $V_{\Phi,\Phi=2\Phi}^{\varkappa,\lambda,\nu}$ $[X,\ Y,\ Z]$ contain exactly one one-dimensional subspaces generated by the suitably normalized zonal polynomial $C_\varkappa\,(X)$ and $\Gamma_\phi^{\varkappa,\lambda,\nu}(X,\ Y,\ Z)$ respectively. The details on the construction of the $C_\phi^{\varkappa,\lambda,\nu}$ will be presented in a subsequent paper. As noted similarly in Davis [6], a representation 2ϕ may occur in (2.2) with multiplicity greater than one for a given $\varkappa$, λ, ν. Hence, the $V_{2\phi}^{\varkappa,\lambda,\nu}$ and the corresponding $\Gamma_\phi^{\varkappa,\lambda,\nu}$ are not uniquely defined when 2ϕ occurs with multiplicity greater than one for a given $\varkappa$, λ, ν.

Table 1

Decompositions of Kronecker products of three irreducible representations (Numbers in parentheses give multiplicities and * indicates invariant representations)

2f	$2\varkappa$	2λ	2ν	Φ						
6	2	2	2	6*	(2)51	(3)42	411	33	(2)321	2^3*
8	4	2	2	8*	(2)71	(3)62*	611	(2)53	(2)521	4^2*
				431	422*					
8	22	2	2	62*	53	(2)521	4^2*	(2)431	(3)422*	4211
				332	3311	(2)3221	2^4*			

Table 1 shows the decompositions of Kronecker products of some low order κ, λ, ν with multiplicities given in parentheses, following the rule for determining the Kronecker products given by Robinson [13, Section 3.3].

The evaluation of integrals in the form $\int_{O(m)} C_\kappa(AH'XH)C_\lambda(BH'YH)C_\nu(CH'ZH)dH$, which also yields a resolution of the non-uniqueness problem mentioned above, follows the direct extension of Davis [6, Section 4]. The utilization of the arguments of James [9, Section 4] and Saw [14] leads to the invariant polynomials $C_\phi^{\kappa,\lambda,\nu}(X, Y, Z)$. These polynomials are linear combinations of $\Gamma_{\phi'}^{\kappa,\lambda,\nu}$ for $\phi'=\phi$ and are '$\Delta_{k,\ell,n}$-orthogonal', and are generated by the set of all distinct products of traces

$$(\mathrm{tr}\, X^{a_1}Y^{b_1}Z^{c_1}X^{d_1}\ldots)^{r1}\ (\mathrm{tr}\, X^{a_2}Y^{b_2}Z^{c_2}X^{d_2}\ldots)^{r2}\ldots$$

of total degrees k, ℓ, n in the elements of X, Y, Z respectively, and provide

$$\int_{O(m)} C_\kappa(AH'XH)C_\lambda(BH'YH)C_\nu(CH'ZH)dH$$
$$=\sum_{\phi\in\kappa\cdot\lambda\cdot\nu} C_\phi^{\kappa,\lambda,\nu}(A, B, C)C_\phi^{\kappa,\lambda,\nu}(X, Y, Z)/C_\phi(I), \qquad (2.3)$$

where $\phi \in \kappa\cdot\lambda\cdot\nu$ signifies that the irreducible representation of $G\ell(m, R)$ indexed by 2ϕ occurs in the decomposition of the Kronecker product $2\kappa \otimes 2\lambda \otimes 2\nu$.

3. Properties of the $C_\phi^{\kappa,\lambda,\nu}$

We give the following results, the proofs of some of which are indicated in Davis [6, Section 3] and omitted here. Throughout this paper, additional subscripts indicating multiplicities, whenever required, are omitted for notational convenience.

3.1. Elementary Properties.

$$C_{\phi}^{\kappa,\lambda,\nu}(X, X, X)=\theta_{\phi}^{\kappa,\lambda,\nu}C_{\phi}(X), \tag{3.1}$$

where $\theta_{\phi}^{\kappa,\lambda,\nu}=C_{\phi}^{\kappa,\lambda,\nu}(I, I, I)/C_{\phi}(I)$.

$$C_{\kappa}^{\kappa,0,0}(X, Y, Z)\overset{\text{def}}{=}C_{\kappa}(X), \tag{3.2}$$

and corresponding results for $C_{\lambda}^{0,\lambda,0}$ and $C_{\nu}^{0,0,\nu}$.

$$\underset{(\sigma\in\kappa\cdot\lambda)}{C_{\sigma}^{\kappa,\lambda,0}}(X, Y, Z)\overset{\text{def}}{=}C_{\sigma}^{\kappa,\lambda}(X, Y), \tag{3.3}$$

and corresponding results for $C_{\sigma}^{\kappa,0,\nu}$ and $C_{\sigma}^{0,\lambda,\nu}$.

$$C_{\phi}^{\kappa,\lambda,\nu}(X, I, I)=[C_{\phi}^{\kappa,\lambda,\nu}(I, I, I)/C_{\kappa}(I)]C_{\kappa}(X), \tag{3.4}$$

and corresponding results for $C_{\phi}^{\kappa,\lambda,\nu}(I, X, I)$ and $C_{\phi}^{\kappa,\lambda,\nu}(I, I, X)$.

$$C_{\phi}^{\kappa,\lambda,\nu}(X, Y, I)=\sum_{\substack{\sigma\in\kappa\cdot\lambda \\ (\phi\in\sigma^{*}\cdot\nu)}}\alpha_{\sigma}^{\kappa,\lambda,\nu;\,\phi}C_{\sigma}^{\kappa,\lambda}(X, Y) \tag{3.5}$$

for a suitable choice of the α where σ^{*} denotes the partition σ ignoring multiplicity, and corresponding results for $C_{\phi}^{\kappa,\lambda,\nu}(X, I, Z)$ and $C_{\phi}^{\kappa,\lambda,\nu}(I, Y, Z)$; in particular,

$$C_{\phi}^{\kappa,\lambda,\nu}(X, X, I)=\sum_{\substack{\sigma\in\kappa\cdot\lambda \\ (\phi\in\sigma^{*}\cdot\nu)}}\alpha_{\sigma}^{\kappa,\lambda,\nu;\,\phi}\theta_{\sigma}^{\kappa,\lambda}C_{\sigma}(X), \tag{3.6}$$

where $\theta_{\sigma}^{\kappa,\lambda}=C_{\sigma}^{\kappa,\lambda}(I, I)/C_{\sigma}(I)$.

$$C_{\phi}^{\kappa,\lambda,\nu}(Y, Y, Z)=\sum_{\sigma^{*}\in\kappa\cdot\lambda}\sum_{\phi'\equiv\phi}\beta_{\sigma^{*};\,\phi'}^{\kappa,\lambda,\nu;\,\phi}C_{\phi'}^{\sigma^{*},\nu}(Y, Z), \tag{3.7}$$

for a suitable choice of the β.

$$C_{\sigma}^{\kappa,\lambda}(X, Y)\ C_{\nu}(Z)=\sum_{\phi\in\sigma^{*}\cdot\nu}\pi_{\sigma,\nu}^{\kappa,\lambda,\nu;\,\phi}C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z), \tag{3.8}$$

for a suitable choice of the π.

$$C_{\phi}^{\kappa,\lambda,\nu}(\alpha X, \beta Y, \gamma Z)=\alpha^{k}\beta^{\ell}\gamma^{n}C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z). \tag{3.9}$$

$$C_\kappa(X)C_\lambda(Y)C_\nu(Z)=\sum_{\phi\in\kappa\cdot\lambda\cdot\nu}\theta_\phi^{\kappa,\lambda,\nu}C_\phi^{\kappa,\lambda,\nu}(X,Y,Z). \tag{3.10}$$

$$C_\kappa(X)C_\lambda(X)C_\nu(X)=\sum_{\phi^*\in\kappa\cdot\lambda\cdot\nu}\sum_{\phi\equiv\phi^*}(\theta_\phi^{\kappa,\lambda,\nu})^2C_{\phi^*}(X). \tag{3.11}$$

3.2. Integrals over O(m).

$$\int_{O(m)}C_\phi^{\kappa,\lambda,\nu}(AH'XH,AH'YH,AH'ZH)dH$$

$$=C_\phi^{\kappa,\lambda,\nu}(X,Y,Z)C_\phi(A)/C_\phi(I). \tag{3.12}$$

$$\int_{O(m)}C_\phi^{\kappa,\lambda,\nu}(A'H'XHA,B,C)dH$$

$$=C_\phi^{\kappa,\lambda,\nu}(A'A,B,C,)C_\kappa(X)/C_\kappa(I). \tag{3.13}$$

$$\int_{O(m)}C_\phi^{\kappa,\lambda,\nu}(A'H'XHA,A'H'YHA,C)dH$$

$$=\sum_{\substack{\sigma\in\kappa\cdot\lambda\\(\phi\in\sigma^*\cdot\nu)}}\sum_{\phi'\equiv\phi}\gamma_{\sigma;\phi'}^{\kappa,\lambda,\nu;\phi}C_{\phi'}^{\sigma^*,\nu}(A'A,C)C_\sigma^{\kappa,\lambda}(X,Y), \tag{3.14}$$

for suitably deflned coefficients γ. (3.13) and (3.14) give the other corresponding results for $C_\phi^{\kappa,\lambda,\nu}(A,B'H'YHB,C)$ and the like.

3.3 Trinomial Expansions.

$$C_\phi(X+Y+Z)=\sum_{\kappa,\lambda,\nu(\phi\in\kappa\cdot\lambda\cdot\nu)}$$

$$\sum_{\phi'\equiv\phi}(f!/k!\,\ell!\,n!)\theta_{\phi'}^{\kappa,\lambda,\nu}C_{\phi'}^{\kappa,\lambda,\nu}(X,Y,Z), \tag{3.15}$$

and, in particular

$$C_f(X+Y+Z)=\sum_{k+\ell+n=f}(f!/k!\,\ell!\,n!)C_f^{k,\ell,n}(X,Y,Z), \tag{3.16}$$

and a similar result for $C_{1^f}(X+Y+Z)$ is derived in terms of $C_{1^f}^{1^k,1^\ell,1^n}(X,Y,Z)$.

$$C_\phi^{\sigma,\nu}(X+Y,Z)=\sum_{\kappa,\lambda(\sigma\in\kappa\cdot\lambda)}\sum_{\phi'\equiv\phi}(s!/k!\,\ell!)\beta_{\sigma;\phi}^{\kappa,\lambda,\nu;\phi'}C_{\phi'}^{\kappa,\lambda,\nu}(X,Y,Z), \tag{3.17}$$

and a corresponding result for $C_{\phi}^{\kappa,\tau}(X, Y+Z)$ is obtained, where the β are given in (3.7).

$$C_{\phi}^{\kappa,\lambda,\nu}(I+A, B, C)/C_{\phi}(I)$$
$$=\sum_{r=0}^{k}\sum_{\rho(\varsigma\in\rho\cdot\lambda\cdot\nu)} b_{\rho,\lambda,\nu;\,\varsigma}^{\kappa,\lambda,\nu;\,\phi} C_{\varsigma}^{\rho,\lambda,\nu}(A, B, C)/C_{\varsigma}(I). \quad (3.18)$$

$$C_{\phi}^{\kappa,\lambda,\nu}(I+A, I+B, C)/C_{\phi}(I)$$
$$=\sum_{r=0}^{k}\sum_{s=0}^{\ell}\sum_{\rho,\sigma(\varsigma\in\rho\cdot\sigma\cdot\nu)} b_{\rho,\sigma,\nu;\,\varsigma}^{\kappa,\lambda,\nu;\,\phi} C_{\varsigma}^{\rho,\sigma,\nu}(A, B, C)/C_{\varsigma}(I). \quad (3.19)$$

The corresponding results for $C_{\phi}^{\kappa,\lambda,\nu}(A, I+B, C)$ and the like for (3.18) and (3.19) hold.

$$C_{\phi}^{\kappa,\lambda,\nu}(I+A, I+B, I+C)/C_{\phi}(I)$$
$$=\sum_{r=0}^{k}\sum_{s=0}^{\ell}\sum_{t=0}^{n}\sum_{\substack{\rho,\sigma,\tau;\,\varsigma \\ (\varsigma\in\rho\cdot\sigma\cdot\tau)}} b_{\rho,\sigma,\tau;\,\varsigma}^{\kappa,\lambda,\nu;\,\phi} C_{\varsigma}^{\rho,\sigma,\tau}(A, B, C)/C_{\varsigma}(I). \quad (3.20)$$

Proof. (3.17) is proved from that $C_{\phi}^{\sigma,\nu}(X+Y, Z)$ is the coefficient of $C_{\phi}^{\sigma,\nu}(A, B)/s!\,n!\,C_{\phi}(I)$ in

$$\int_{O(m)} \mathrm{etr}[AH'(X+Y)H+BH'ZH]dH$$
$$=\sum_{\kappa,\lambda,\nu;\,\phi}^{\infty} C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z) C_{\phi}^{\kappa,\lambda,\nu}(A, A, B)/k!\,\ell!\,n!\,C_{\phi}(I)$$
$$=\sum_{\kappa,\lambda,\nu;\,\phi}^{\infty} C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z)\sum_{\sigma^{*}\in\kappa\cdot\lambda}\sum_{\phi'\equiv\phi}\beta_{\sigma^{*};\,\phi'}^{\kappa,\lambda,\nu;\,\phi}$$
$$C_{\phi'}^{\sigma^{*},\nu}(A, B)/k!\,\ell!\,n!\,C_{\phi}(I). \qquad (\because\ (3.7))$$

(3.18)−(3.20) are shown by a similar method to Davis [5, Eq. (6.6)].

4. Complete and incomplete gamma and beta-type integrals

We shall show several Laplace transforms and beta-type integrals of the $C_\phi^{\kappa,\lambda,\nu}$. Incomplete gamma and beta functions are also established.

4.1. Laplace Transforms.

$$\int_{R>0} \operatorname{etr}(-WR)|R|^{a-p} C_\phi^{\kappa,\lambda,\nu}(AR, BR, CR)dR$$
$$=\Gamma_m(a, \phi)|W|^{-a} C_\phi^{\kappa,\lambda,\nu}(AW^{-1}, BW^{-1}, CW^{-1}), \quad (4.1)$$

and in particular

$$E_V C_\phi^{\kappa,\lambda,\nu}(V'AV, V'BV, V'CV)$$
$$= 2^f (\tfrac{1}{2}n)_\phi C_\phi^{\kappa,\lambda,\nu}(W'AW, W'BW, W'CW), \quad (4.2)$$

where $VV' \sim W_m(n, \Sigma)$ and $WW' = \Sigma$, and $p = (m+1)/2$.

$$\int_{R>0} \operatorname{etr}(-WR)|R|^{a-p} C_\phi^{\kappa,\lambda,\nu}(ARA', B, C)dR$$
$$=\Gamma_m(a, \kappa)|W|^{-a} C_\phi^{\kappa,\lambda,\nu}(AW^{-1}A', B, C). \quad (4.3)$$

$$\int_{R>0} \operatorname{etr}(-WR)|R|^{a-p} C_\phi^{\kappa,\lambda,\nu}(AR^{-1}, BR^{-1}, CR^{-1})dR$$
$$=\Gamma_m(a, -\phi)|W|^{-a} C_\phi^{\kappa,\lambda,\nu}(AW, BW, CW), \quad (4.4)$$

where

$$\Gamma_m(a, -\phi) = (-1)^f \ \Gamma_m(a)/(-a+p)_\phi. \quad (4.5)$$

$$\int_{R>0} \operatorname{etr}(-WR)|R|^{a-p} C_\phi^{\kappa,\lambda,\nu}(AR^{-1}A', B, C)dR$$
$$=\Gamma_m(a, -\kappa)|W|^{-a} C_\phi^{\kappa,\lambda,\nu}(AWA', B, C). \quad (4.6)$$

proof. These Laplace transforms are consequences of (2.3). Use is made of Constantine [4, Eq. (10)] as for the proof of (4.4) and (4.6).

4.2. Beta-type Integrals.

$$\int_0^I\int_0^I_{0<S+T<I} |S|^{a-p}|T|^{b-p}|I-S-T|^{c-p}\, C_\phi^{\varkappa,\lambda,\nu}(S, T, I-S-T)\,dSdT$$
$$=\Gamma_m(a, \varkappa)\Gamma_m(b, \lambda)\Gamma_m(c, \nu)[\Gamma_m(a+b+c, \phi)]^{-1}\,\theta_\phi^{\varkappa,\lambda,\nu}\, C_\phi(I). \qquad (4.7)$$

$$\int_0^I\int_0^I\int_0^I_{0<S+T+U<I} |S|^{a-p}|T|^{b-p}|U|^{c-p}|I-S-T-U|^{d-p}$$
$$C_\phi^{\varkappa,\lambda,\nu}(S, T, U)\,dSdTdU=\Gamma_m(a, \varkappa)\Gamma_m(b, \lambda)$$
$$\Gamma_m(c, \nu)\Gamma_m(d)\,\theta_\phi^{\varkappa,\lambda,\nu}C_\phi(I)/\Gamma_m(a+b+c+d, \phi). \qquad (4.8)$$

$$\int_0^I |S|^{a-p}|I-S|^{b-p}\, C_\phi^{\varkappa,\lambda,\nu}(AS, BS, CS)\,dS$$
$$=\Gamma_m(a, \phi)\Gamma_m(b)[\Gamma_m(a+b, \phi)]^{-1}\, C_\phi^{\varkappa,\lambda,\nu}(A, B, C). \qquad (4.9)$$

$$\int_0^I |S|^{a-p}|I-S|^{b-p}\, C_\phi^{\varkappa,\lambda,\nu}(ASA', B, C)\,dS$$
$$=\Gamma_m(a, \varkappa)\Gamma_m(b)[\Gamma_m(a+b, \varkappa)]^{-1}\, C_\phi^{\varkappa,\lambda,\nu}(AA', B, C). \qquad (4.10)$$

$$\int_0^I |S|^{a-p}|I-S|^{b-p}\, C_\phi^{\varkappa,\lambda,\nu}(AS^{-1}, BS^{-1}, CS^{-1})\,dS$$
$$=\Gamma_m(a, -\phi)\Gamma_m(b)[\Gamma_m(a+b, -\phi)]^{-1}\, C_\phi^{\varkappa,\lambda,\nu}(A, B, C). \qquad (4.11)$$

$$\int_0^I |S|^{a-p}|I-S|^{b-p}\, C_\phi^{\varkappa,\lambda,\nu}(AS^{-1}A', B, C)\,dS$$
$$=\Gamma_m(a, -\varkappa)\Gamma_m(b)\Gamma_m(a+b, -\varkappa)]^{-1}\, C_\phi^{\varkappa,\lambda,\nu}(AA', B, C). \qquad (4.12)$$

Proof. To prove (4.7) we first evaluate

$$\Gamma=\int_{X>0}\int_{Y>0}\int_{Z>0} \mathrm{etr}-(X+Y+Z)\,|X|^{a-p}|Y|^{b-p}|Z|^{c-p}$$
$$C_\phi^{\varkappa,\lambda,\nu}(X, Y, Z)\,dXdYdZ. \qquad (4.13)$$

Making the transformations $R=X+Y+Z$, $S=R^{-1/2}XR^{-1/2}$, $T=R^{-1/2}YR^{-1/2}$ and using (4.1) gives

$$\Gamma=\Gamma_m(a+b+c,\ \phi)\Lambda, \tag{4.14}$$

where Λ is the left hand side of (4.7). Now, in general, we can show by using (2.3) and (4.3) that

$$\int_{X>0}\int_{Y>0}\int_{Z>0} \operatorname{etr}-(AX+BY+CZ)|X|^{a-p}|Y|^{b-p}|Z|^{c-p}$$
$$C_\phi^{\kappa,\lambda,\nu}(X,\ Y,\ Z)dXdYdZ=\Gamma_m(a,\ \kappa)\Gamma_m(b,\ \lambda)\ \Gamma_m(c,\ \nu)$$
$$[|A|^a|B|^b|C|^c]^{-1}\ C_\phi^{\kappa,\lambda,\nu}(A^{-1},\ B^{-1},\ C^{-1}),$$

leading to

$$\Gamma=\Gamma_m(a,\ \kappa)\Gamma_m(b,\ \lambda)\Gamma_m(c,\ \nu)\ \theta_\phi^{\kappa,\lambda,\nu}\ C_\phi(I). \tag{4.15}$$

(4.14) and (4.15) establish the required result (4.7).

(4.8) is shown by evaluating the integral form

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\int_{V>0} \operatorname{etr}-(X+Y+Z+V)|X|^{a-p}|Y|^{b-p}|Z|^{c-p}|V|^{d-p}$$
$$C_\phi^{\kappa,\lambda,\nu}(X,\ Y,\ Z)dXdYdZdV,$$

making the transformations $R=X+Y+Z+V$, $S=R^{-1/2}XR^{-1/2}$, $T=R^{-1/2}YR^{-1/2}$, $U=R^{-1/2}ZR^{-1/2}$.

(4.9) and (4.10) are proved using (2.3) and Constantine [3, Eq. (22)]. (4.11) and (4.12) are established by using (2.3) and Khatri [11, Eq. (17)].

4.3. Incomplete Gamma Functions.

$$\int_0^X \operatorname{etr}(-CR)|R|^{a-p}\,C_\kappa(AR)C_\lambda(BR)dR$$
$$=\Gamma_m(p)|X|^a\sum_{n=0}^{\infty}\sum_{\nu;\ \phi\in\kappa\cdot\lambda\cdot\nu}\Gamma_m(a,\ \phi)\,\theta_\phi^{\kappa,\lambda,\nu}$$

$$[n!\,\Gamma_m(a+p,\phi)]^{-1} C_\phi^{\varkappa,\lambda,\nu}(AX, BX, -CX). \quad (4.16)$$

$$\int_0^X \operatorname{etr}(-CR)\,|R|^{a-p}\, C_\sigma^{\varkappa,\lambda}(AR, BR)\,dR$$

$$=\Gamma_m(p)\,|X|^a \sum_{n=0}^{\infty}\sum_{\nu;\,\phi\in\sigma^*\cdot\nu}\Gamma_m(a,\phi)\,\pi_{\phi,\nu}^{\varkappa,\lambda,\nu;\,\sigma}$$

$$[n!\,\Gamma_m(a+p,\phi)]^{-1} C_\phi^{\varkappa,\lambda,\nu}(AX, BX, -CX). \quad (4.17)$$

4.4. Incomplete Beta Functions.

$$\int_0^X |S|^{a-p}|I-CS|^{b-p}\, C_\varkappa(AS)\, C_\lambda(BS)\,dS$$

$$=\Gamma_m(p)\,|X|^a \sum_{n=0}^{\infty}\sum_{\nu;\,\phi\in\varkappa\cdot\lambda\cdot\nu}\Gamma_m(a,\phi)\;(p-b)_\nu\,\theta_\phi^{\varkappa,\lambda,\nu}$$

$$[n!\,\Gamma_m(a+p,\phi)]^{-1} C_\phi^{\varkappa,\lambda,\nu}(AX, BX, CX). \quad (4.18)$$

$$\int_0^X |S|^{a-p}|I-CS|^{b-p}\, C_\sigma^{\varkappa,\lambda}(AS, BS)\,dS$$

$$=\Gamma_m(p)\,|X|^a \sum_{n=0}^{\infty}\sum_{\nu;\,\phi\in\sigma^*\cdot\nu}\Gamma_m(a,\phi)\;(p-b)_\nu\,\pi_{\sigma,\nu}^{\varkappa,\lambda,\nu;\,\phi}$$

$$[n!\,\Gamma_m(a+p,\phi)]^{-1} C_\phi^{\varkappa,\lambda,\nu}(AX, BX, CX). \quad (4.19)$$

$$\int_0^X |S|^{a-p}\, C_\phi^{\varkappa,\lambda,\nu}(AS, BS, CS)\,dS$$

$$=\Gamma_m(p)\Gamma_m(a,\phi)[\Gamma_m(a+p,\phi)]^{-1}\,|X|^a\, C_\phi^{\varkappa,\lambda,\nu}(AX, BX, CX). \quad (4.20)$$

$$\int_0^X |S|^{a-p}\, C_\phi^{\varkappa,\lambda,\nu}(ASA', B, C)\,dS$$

$$=\Gamma_m(p)\Gamma_m(a,\varkappa)[\Gamma_m(a+p,\varkappa)]^{-1}\,|X|^a\, C_\phi^{\varkappa,\lambda,\nu}(AXA', B, C). \quad (4.21)$$

Proof. (4.18) and (4.19) are shown as extensions of Davis [5, Eq. (3.3)]. (4.20) and (4.21) are proved by applying Constantine [3, Eq. (22)], with the use of (3.12) and (2.3) respectively.

5. Generalized Laguerre polynomials with three matrix arguments

We shall show some generalizations of Laguerre polynomials, having three matrix arguments, along the lines of Davis [5].

5.1. Definition I.

Define

$$L^{t,u,w}_{\kappa,\lambda,\nu;\phi}(X, Y, Z) = \mathrm{etr}(X+Y+Z) \int_{R>0}\int_{S>0}\int_{T>0} \mathrm{etr}(-R-S-T)$$

$$|R|^t |S|^u |T|^w C^{\kappa,\lambda,\nu}_{\phi}(R, S, T) A_t(RX) A_u(SY)$$

$$A_w(TZ)\, dR dS dT, \qquad (5.1)$$

where A_t is the Bessel function of Herz [8].

Laplace transform.

$$\int_{R>0}\int_{S>0}\int_{T>0} \mathrm{etr}(-RX-SY-TZ) |R|^t |S|^u |T|^w$$

$$L^{t,u,w}_{\kappa,\lambda,\nu;\phi}(R, S, T)\, dR dS dT = \Gamma_m(t+p, \kappa)\Gamma_m(u+p, \lambda)$$

$$\Gamma_m(w+p, \nu) |X|^{-t-p} |Y|^{-u-p} |Z|^{-w-p}$$

$$C^{\kappa,\lambda,\nu}_{\phi}(I-X^{-1}, I-Y^{-1}, I-Z^{-1}). \qquad (5.2)$$

Serial expression.

$$L^{t,u,w}_{\kappa,\lambda,\nu;\phi}(R, S, T) = (t+p)_{\kappa} (u+p)_{\lambda} (w+p)_{\nu}\, C_{\phi}(I)$$

$$\Sigma^{k}_{r=0}\Sigma^{\ell}_{s=0}\Sigma^{n}_{t=0}\Sigma_{\rho,\sigma,\tau;\zeta}(-1)^{r+s+t} b^{\kappa,\lambda,\nu;\phi}_{\rho,\sigma,\tau;\zeta}$$

$$C^{\rho,\sigma,\tau}_{\zeta}(R, S, T)/(t+p)_{\rho} (u+p)_{\sigma} (w+p)_{\tau} C_{\zeta}(I), \qquad (5.3)$$

where the b are given by (3.20).

Generating function.

$$|I-R|^{-t-p}|I-S|^{-u-p}|I-T|^{-w-p}\int_{O(m)} \mathrm{etr}\{-XH'R(I-R)^{-1}H$$

$$-YH'S(I-S)^{-1}H-ZH'T(I-T)^{-1}H\}dH$$

$$=\Sigma^{\infty}_{\varkappa,\lambda,\nu;\phi} L^{t,u,w}_{\varkappa,\lambda,\nu;\phi}(X, Y, Z)\ C^{\varkappa,\lambda,\nu}_{\phi}(R, S, T)$$

$$/k!\ \ell!\ n!\ C_{\phi}(I). \tag{5.4}$$

A relation with Khatri's polynomial (See Khatri [12]) is

$$\int_{O(m)} L^{t}_{\varkappa}(HAH', X)\ L^{u}_{\lambda}(HBH', Y)\ L^{w}_{\nu}(HCH', Z)dH$$

$$=\Sigma_{\phi\in\varkappa\cdot\lambda\cdot\nu}C^{\varkappa,\lambda,\nu}_{\phi}(X, Y, Z)\ L^{t,u,w}_{\varkappa,\lambda,\nu;\phi}(A, B, C)/C_{\phi}(I). \tag{5.5}$$

5.2. Definition II.

Define

$$L^{t,u}_{\varkappa,\lambda,\nu;\phi}(X, Y, Z)=\mathrm{etr}(X+Y)$$

$$\int_{R>0}\int_{S>0} \mathrm{etr}(-R-S)|R|^{t}\ |S|^{u}\ C^{\varkappa,\lambda,\nu}_{\phi}(R, S, Z)$$

$$A_{t}\ (RX)\ A_{u}\ (SY)\ dRdS. \tag{5.6}$$

Laplace transform.

$$\int_{R>0}\int_{S>0} \mathrm{etr}(-RX\ -SY)|R|^{t}\ |S|^{u}\ L^{t,u}_{\varkappa,\lambda,\nu;\phi}(R, S, Z)dRdS$$

$$=\Gamma_m(t+p, \varkappa)\Gamma_m(u+p, \lambda)|X|^{-t-p}|Y|^{-u-p}$$

$$C^{\varkappa,\lambda,\nu}_{\phi}(I-X^{-1}, I-Y^{-1}, Z). \tag{5.7}$$

Serial expression.

$$L^{t,u}_{\varkappa,\lambda,\nu;\phi}(R, S, T)=(t+p)_{\varkappa}\ (u+p)_{\lambda}\ C_{\phi}(I)$$

$$\sum_{r=0}^{k}\sum_{s=0}^{\ell}\sum_{\rho,\sigma(\zeta\in\rho\cdot\sigma\cdot\nu)}(-1)^{r+s}$$

$$b_{\rho,\sigma,\nu;\,\zeta}^{\kappa,\lambda,\nu;\,\phi} C_{\zeta}^{\rho,\sigma,\nu}(R,\ S,\ T)/(t+p)_{\rho}\ (u+p)_{\sigma}\ C_{\zeta}(I), \qquad (5.8)$$

where the b are given by (3.19).

Generating function.

$$|I-R|^{-t-p}|I-S|^{-u-p}\int_{O(m)}$$

$$\mathrm{etr}\{-XH'R(I-R)^{-1}H-YH'S(I-R)^{-1}H\}\ \mathrm{etr}(Z)dH$$

$$=\sum_{\kappa,\lambda,\nu;\,\phi}^{\infty} L_{\kappa,\lambda,\nu;\,\phi}^{t,u}(X,\ Y,\ Z)\ C_{\phi}^{\kappa,\lambda,\nu}(R,\ S,\ I)/k!\ \ell!\ n!\ C_{\phi}(I). \qquad (5.9)$$

5.3. Definition III.

Define

$$L_{\kappa,\lambda,\nu;\,\phi}^{t}(X,\ Y,\ Z)=\mathrm{etr}(X)\int_{R>0}\mathrm{etr}(-R)|R|^{t}$$

$$C_{\phi}^{\kappa,\lambda,\nu}(R,\ Y,\ Z)A_t\ (RX)dR. \qquad (5.10)$$

Laplace transform.

$$\int_{R>0}\mathrm{etr}(-RX)|R|^{t}\ L_{\kappa,\lambda,\nu;\,\phi}^{t}(R,\ Y,\ Z)dR$$

$$=\Gamma_m(t+p,\ \kappa)|X|^{-t-p}\ C_{\phi}^{\kappa,\lambda,\nu}(I-X^{-1},\ Y,\ Z). \qquad (5.11)$$

Serial expression.

$$L_{\kappa,\lambda,\nu;\,\phi}^{t}(R,\ S,\ T)=(t+p)_{\kappa}\ C_{\phi}(I)\ \sum_{r=0}^{k}\sum_{\rho(\zeta\in\rho\cdot\lambda\cdot\nu)}$$

$$(-1)^{r}\ b_{\rho,\lambda,\nu;\,\zeta}^{\kappa,\lambda,\nu;\,\phi} C_{\zeta}^{\rho,\lambda,\nu}(R,\ S,\ T)/(t+p)_{\rho}\ C_{\zeta}(I), \qquad (5.12)$$

where the b are given by (3.18).

Generating function.

$$|I-R|^{-t-p}\int_{O(m)} \operatorname{etr}\{-XH'R(I-R)^{-1}H\} \operatorname{etr}(Y+Z)dH$$

$$=\Sigma^{\infty}_{\kappa,\lambda,\nu;\,\phi} L^{t}_{\kappa,\lambda,\nu;\,\phi}(X, Y, Z)C^{\kappa,\lambda,\nu}_{\phi}(R, I, I)/k!\,\ell!\,n!\,C_{\phi}(I). \qquad (5.13)$$

The above Laguerre polynomials have the orthogonality property.

6. Some other expansions

The following expansions are derived along the lines of Davis [5, Section 6]. Multiplying the both sides of

$$\int_{O(m)} \operatorname{etr}(AH'XH+BH'YH+CH'ZH)dH$$

$$=\Sigma^{\infty}_{\kappa,\lambda,\nu;\,\phi} C^{\kappa,\lambda,\nu}_{\phi}(A, B, C)C^{\kappa,\lambda,\nu}_{\phi}(X, Y, Z)/k!\,\ell!\,n!\,C_{\phi}(I),$$

by $\operatorname{etr}(-X)$, $\operatorname{etr}(-X-Y)$ and $\operatorname{etr}(-X-Y-Z)$, we obtain

$$\operatorname{etr}(-X)\ \Sigma^{\infty}_{\kappa,\lambda,\nu;\,\phi} C^{\kappa,\lambda,\nu}_{\phi}(A, B, C)C^{\kappa,\lambda,\nu}_{\phi}(X, Y, Z)/k!\,\ell!\,n!\,C_{\phi}(I)$$

$$=\Sigma^{\infty}_{\kappa,\lambda,\nu;\,\phi} C^{\kappa,\lambda,\nu}_{\phi}(A-I, B, C)C^{\kappa,\lambda,\nu}_{\phi}(X, Y, Z)$$

$$/k!\,\ell!\,n!\,C_{\phi}(I), \qquad (6.1.1)$$

$$\operatorname{etr}(-X-Y)\ \Sigma^{\infty}_{\kappa,\lambda,\nu;\,\phi} C^{\kappa,\lambda,\nu}_{\phi}(A, B, C)C^{\kappa,\lambda,\nu}_{\phi}(X, Y, Z)$$

$$/k!\,\ell!\,n!\,C_{\phi}(I)=\Sigma^{\infty}_{\kappa,\lambda,\nu;\,\phi} C^{\kappa,\lambda,\nu}_{\phi}(A-I, B-I, C)$$

$$C^{\kappa,\lambda,\nu}_{\phi}(X, Y, Z)/k!\,\ell!\,n!\,C_{\phi}(I), \qquad (6.1.2)$$

$$\operatorname{etr}(-X-Y-Z)\ \Sigma^{\infty}_{\kappa,\lambda,\nu;\,\phi} C^{\kappa,\lambda,\nu}_{\phi}(A, B, C)C^{\kappa,\lambda,\nu}_{\phi}(X, Y, Z)/$$

$$k!\,\ell!\,n!\,C_{\phi}(I)=\Sigma^{\infty}_{\kappa,\lambda,\nu;\,\phi} C^{\kappa,\lambda,\nu}_{\phi}(A-I, B-I, C-I)$$

$$C^{\kappa,\lambda,\nu}_{\phi}(X, Y, Z)/k!\,\ell!\,n!\,C_{\phi}(I). \qquad (6.1.3)$$

These yield the following

$$|I+X|^{-a}\sum_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}C_{\phi}^{\kappa,\lambda,\nu}(A,B,C)C_{\phi}^{\kappa,\lambda,\nu}(X(I+X)^{-1}, Y(I+Y)^{-1}, Z(I+Z)^{-1})/k!\,\ell!\,n!\,C_{\phi}(I)$$

$$=\sum_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}C_{\phi}^{\kappa,\lambda,\nu}(A-I,B,C)\,C_{\phi}^{\kappa,\lambda,\nu}(X,Y,Z)/k!\,\ell!\,n!\,C_{\phi}(I), \qquad (6.2.1)$$

$$|I+X+Y|^{-a}\sum_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}C_{\phi}^{\kappa,\lambda,\nu}(A,B,C)\,C_{\phi}^{\kappa,\lambda,\nu}(X(I+X+Y)^{-1}, Y(I+X+Y)^{-1}, Z(I+X+Y)^{-1})/k!\,\ell!\,n!\,C_{\phi}(I)=\sum_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}C_{\phi}^{\kappa,\lambda,\nu}(A-I,B-I,C)\,C_{\phi}^{\kappa,\lambda,\nu}(X,Y,Z)/k!\,\ell!\,n!\,C_{\phi}(I), \qquad (6.2.2)$$

$$|I+X+Y+Z|^{-a}\sum_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}C_{\phi}^{\kappa,\lambda,\nu}(A,B,C)\,C_{\phi}^{\kappa,\lambda,\nu}(X(I+X+Y+Z)^{-1}, Y(I+X+Y+Z)^{-1}, Z(I+X+Y+Z)^{-1})/k!\,\ell!\,n!\,C_{\phi}(I)$$

$$=\sum_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}C_{\phi}^{\kappa,\lambda,\nu}(A-I,B-I,C-I)\,C_{\phi}^{\kappa,\lambda,\nu}(X,Y,Z)/k!\,\ell!\,n!\,C_{\phi}(I). \qquad (6.2.3)$$

Also, we can derive

$$\mathrm{etr}(-X)\int_{O(m)}\mathrm{etr}(AH'XH+CH'ZH)\,{}_0F_1(u;BH'YH)dH$$

$$=\sum_{\kappa,\lambda,\nu;\phi}^{\infty}C_{\phi}^{\kappa,\lambda,\nu}(A-I,B,C)C_{\phi}^{\kappa,\lambda,\nu}(X,Y,Z)/k!\,\ell!\,n!\,(u)_{\lambda}C_{\phi}(I), \qquad (6.3.1)$$

$$\mathrm{etr}(-X-Y)\int_{O(m)}\mathrm{etr}(AH'XH+BH'YH)\,{}_0F_1(w;CH'ZH)dH$$

$$=\Sigma_{\kappa,\lambda,\nu;\phi}^{\infty} C_{\phi}^{\kappa,\lambda,\nu}(A-I, B-I, C)C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z)/$$

$$k!\,\ell!\,n!\,(w)_{\nu}C_{\phi}(I), \quad (6.3.2)$$

$$\mathrm{etr}(-X-Y-Z)\int_{O(m)}\mathrm{etr}(AH'XH+BH'YH+CH'ZH)dH$$

$$=\Sigma_{\kappa,\lambda,\nu;\phi}^{\infty} C_{\phi}^{\kappa,\lambda,\nu}(A-I, B-I, C-I)C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z)/$$

$$k!\,\ell!\,n!\,C_{\phi}(I), \quad (6.3.3)$$

$$\mathrm{etr}(-X, -Y-Z)\int_{O(m)} {}_0F_1(t;AH'XH)\,{}_0F_1(u;BH'YH)$$

$${}_0F_1(w;CH'ZH)dH=\Sigma_{\kappa,\lambda,\nu;\phi}^{\infty}(-1)^{f}\,L_{\kappa,\lambda,\nu}^{t-p,u-p,w-p}(A, B, C)$$

$$C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z)/k!\,\ell!\,n!\,(t)_{\kappa}(u)_{\lambda}(w)_{\nu}C_{\phi}(I).$$

$$(6.3.4)$$

The following identities hold:

$$\Sigma_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda,\nu}C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z)/k!\,\ell!\,n!$$

$$=|I+W|^{-a}\Sigma_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda,\nu}C_{\phi}^{\kappa,\lambda,\nu}((X+W)(I+W)^{-1},$$

$$Y(I+W)^{-1}, Z(I+W)^{-1})/k!\,\ell!\,n!, \quad (6.4.1)$$

$$\Sigma_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda,\nu}C_{\phi}^{\kappa,\lambda,\nu}(X, Y, Z)/k!\,\ell!\,n!\,(u)_{\lambda}(w)_{\nu}$$

$$=|I+W|^{-a}\Sigma_{\kappa,\lambda,\nu;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda,\nu}$$

$$C_{\phi}^{\kappa,\lambda,\nu}((X+W)(I+W)^{-1}, Y(I+W)^{-1},$$

$$Z(I+W)^{-1})/k!\,\ell!\,n!\,(u)_{\lambda}(w)_{\nu}. \quad (6.4.2)$$

7. Some useful differential identities

We shall derive some differential identities satisfied by the $C_{\phi}^{\kappa,\lambda,\nu}$; the corresponding results for the zonal polynomials are obtained by Fujikoshi [7]. Putting $a=\frac{1}{2}n$, $A=(2/n)I$, $B=(2/n)Y$, $C=(2/n)Z$, $W=X^{-1}$ in (4.1) yields that

$$[\Gamma_m(\tfrac{1}{2}n)|X|^{\frac{1}{2}n}]^{-1}\int_{R>0}\mathrm{etr}(-X^{-1}R)|R|^{\frac{1}{2}n-p}(2/n)^f$$

$$C_\phi^{\kappa,\lambda,\nu}(R, YR, ZR)dR=(2/n)^f(n/2)_\phi C_\phi^{\kappa,\lambda,\nu}(X, YX, ZX)$$

$$=[1+(1/n)a_1(\phi)+(1/6n^2)\{3a_1(\phi)^2-a_2(\phi)+f\}$$

$$+O(n^{-3})]C_\phi^{\kappa,\lambda,\nu}(X, YX, ZX)$$

$$=[1+(1/n)\mathrm{tr}(X\partial)^2+(1/6n^2)\{3(\mathrm{tr}(X\partial)^2)^2+8\,\mathrm{tr}(X\partial)^3\}$$

$$+O(n^{-3})]C_\phi^{\kappa,\lambda,\nu}(\Sigma, Y\Sigma, Z\Sigma)|_{\Sigma=X} \quad (7.1)$$

Here, $a_1(\phi)=\sum_{\alpha=1}^m f_\alpha(f_\alpha-\alpha)$,

$a_2(\phi)=\sum_{\alpha=1}^m f_\alpha(4f_\alpha^2-6\alpha f_\alpha+3\alpha^2)$ for the partition $\phi=(f_1, f_2, \cdots, f_m)$, $\partial=(\partial_{ij})$, with $\partial_{ij}=\frac{1}{2}(1+\delta_{ij})\partial/\partial\sigma_{ij}$ and $\Sigma=(\sigma_{ij})$, is the matrix of differential operators, and the last identity in (7.1) is obtained from the Taylor series expansion method (e.g., Sugiura and Fujikoshi [15]). Comparing the coefficients of the terms $1/n$ and $1/n^2$ in (7.1) gives

$$a_1(\phi)C_\phi^{\kappa,\lambda,\nu}(X, YX, ZX)$$

$$=\mathrm{tr}(X\partial)^2C_\phi^{\kappa,\lambda,\nu}(\Sigma, Y\Sigma, Z\Sigma)|_{\Sigma=X}, \quad (7.2.1)$$

$$\{3a_1(\phi)^2-a_2(\phi)+f\}C_\phi^{\kappa,\lambda,\nu}(X, YX, ZX)$$

$$=\{3(\mathrm{tr}(X\partial)^2)^2+8\mathrm{tr}(X\partial)^3\}C_\phi^{\kappa,\lambda,\nu}(\Sigma, Y\Sigma, Z\Sigma)|_{\Sigma=X} \quad (7.2.2)$$

Similarly from (4.3), we obtain

$$a_1(\phi)C_\phi^{\kappa,\lambda,\nu}(X, Y, Z)=\mathrm{tr}(X\partial)^2C_\phi^{\kappa,\lambda,\nu}(\Sigma, Y, Z)|_{\Sigma=X}, \quad (7.3.1)$$

$$\{3a_1(\kappa)^2-a_2(\kappa)+k\}C_\phi^{\kappa,\lambda,\nu}(X, Y, Z)$$

$$=\{3(\mathrm{tr}(X\partial)^2)^2+8\mathrm{tr}(X\partial)^3\}C_\phi^{\kappa,\lambda,\nu}(\Sigma, Y, Z)|_{\Sigma=X}. \qquad (7.3.2)$$

The corresponding results for $\Sigma=Y$ and $\Sigma=Z$ for (7.2) and (7.3) hold.

(7.3) leads to the following simultaneous differential identities, with $\partial_\ell=(\partial_{\ell,ij})$ with $\partial_{\ell,ij}=½(1+\delta_{ij})\partial/\partial\sigma_{\ell,ij}$, $\Sigma_\ell=(\sigma_{\ell,ij})$, $\ell=1, 2, 3$:

$$a_1(\kappa)a_1(\lambda)C_\phi^{\kappa,\lambda,\nu}(X, Y, Z)$$

$$=\mathrm{tr}(X\partial_1)^2\,\mathrm{tr}(Y\partial_2)^2\,C_\phi^{\kappa,\lambda,\nu}(\Sigma_1, \Sigma_2, Z)|_{\Sigma_1=X, \Sigma_2=Y},$$

$$(7.4.1)$$

$$a_1(\kappa)\{3a_1(\lambda)^2-a_2(\lambda)+\ell\}C_\phi^{\kappa,\lambda,\nu}(X, Y, Z)=\mathrm{tr}(X\partial_1)^2$$

$$\{3(\mathrm{tr}(Y\partial_2)^2)^2+8\mathrm{tr}(Y\partial_2)^3\}C_\phi^{\kappa,\lambda,\nu}(\Sigma_1, \Sigma_2, Z)|_{\Sigma_1=X, \Sigma_2=Y}, \qquad (7.4.2)$$

and

$$a_1(\kappa)a_1(\lambda)a_1(\nu)C^{\kappa,\lambda,\nu}(X, Y, Z)$$

$$=\mathrm{tr}(X\partial_1)^2\,\mathrm{tr}(Y\partial_2)^2\,\mathrm{tr}(Z\partial_3)^2$$

$$C_\phi^{\kappa,\lambda,\nu}(\Sigma_1, \Sigma_2, \Sigma_3)|_{\Sigma_1=X, \Sigma_2=Y, \Sigma_3=Z}, \qquad (7.5.1)$$

$$a_1(\kappa)a_1(\lambda)\{3a_1(\nu)^2-a_2(\nu)+n\}C_\phi^{\kappa,\lambda,\nu}(X, Y, Z)$$

$$=\mathrm{tr}(X\partial_1)^2\,\mathrm{tr}(Y\partial_2)^2\{3(\mathrm{tr}(Z\partial_3)^2)^2+8\mathrm{tr}(Z\partial_3)^3\}$$

$$C_\phi^{\kappa,\lambda,\nu}(\Sigma_1, \Sigma_2, \Sigma_3)|_{\Sigma_1=X, \Sigma_2=Y, \Sigma_3=Z}. \qquad (7.5.2)$$

The other possible results corresponding to (7.4) and (7.5) are readily obtained.

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