Estimation and Testing for the SUR Model with Random Walk and Deterministic Trend

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abstract

We consider the 2-equations seemingly unrelated regression (SUR) model. A random walk model and a deterministic trend model are mixed. We derive the asymptotic distributions of the OLS and the restricted and unrestricted SUR estimators using nonstandard asymptotics. Small sample properties are examined by Monte Carlo experiment for a 2-equation system. The experiments show that the SUR estimators are more efficient than the OLS estimator and these estimators are more powerful than the OLS estimator as a unit root test statistics.

1. Introduction

Maekawa and Hisamatsu(2002) considered the seemingly unrelated regression (SUR) models with I(1) explanatory variables. The two cases were analyzed:(I) explanatory variables are integrated of order one and (II) explanatory variables are the lagged dependent variables with a coefficient of unity in all or some equations. Non-standard asymptotic distributions of the OLS and SUR estimators were derived. It was shown that SUR estimator is more efficient than the OLS estimator and a unit root test based on the SUR estimator is more powerful than a test based on the OLS estimator in the case (II). Furthermore it was considered a cointegration test in the case (I).

The present paper extends the study of Maekawa and Hisamatsu(2002) to the more general situation that the stationary and nonstationary processes are mixed in the SUR model. We consider the

SUR models in which a random walk model and a deterministic trend model are mixed. We derive non-standard asymptotic distributions of the OLS and SUR estimators and show the order of the consistency of the estimators. Relative efficiency of these estimators in small sample is examined by Monte Carlo experiment. The experiment shows that the restricted and unrestricted SUR estimators are more efficient than the OLS estimator. In this model we also show that a unit root test based on the SUR estimator is more powerful than a test based on the OLS estimator.

This paper is organized as follows. In section 2 we present the model and definition of the OLS and SUR estimators. In section 3 we derive the asymptotic distributions of these estimators by using nonstandard asymptotics. In section 4 we examine the small sample properties of these statistics by Monte Carlo experiments. We also calculate the power of these estimators as a unit root test statistics and compare the performance. Finally in section 5 we provide some concluding remarks.

2. SUR model with random walk and deterministic trend

We consider the following 2-equation SUR model

$$\begin{aligned} y_t &= \rho y_{t-1} + v_t, \\ x_t &= c + \beta \cdot t + w_t \end{aligned} \qquad (1)$$

where

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} \sim N\left(0, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\right),$$

and true value of ρ is unity.

To write the system compactly we often use a notation $A = (a_{ij})$ for a matrix of an element a_{ij} and introduce the following vectors and matrices

$$x_1 = (\mathrm{y}_{-1}), \quad x_2 = (\mathbf{i} \quad \mathbf{t}), \quad \beta' = (\rho, c, \beta)', \quad u_1 = v, \quad u_2 = w$$

where all components of $\mathbf{i}$ are unity and those of $\mathbf{t}$ are deterministic time trend. Then we can write the system as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \beta + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{2}$$

or more compactly

$$Y = X\beta + U \tag{3}$$

where the definitions of Y, X and U are self-evident. The distribution of U can be written as

$$U \sim N(0, \Omega)$$

where

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} I & \sigma_{12} I \\ \sigma_{21} I & \sigma_{22} I \end{pmatrix} = \Sigma_{\mathrm{U}} \otimes I,$$

$\Sigma_U = (\sigma_{ij})$ is assumed positive definite and $\otimes$ signifies the Kronecker product.

The OLS and SUR estimators of β are written as

$$OLS: \quad \hat{\beta} = (X'X)^{-1} X'Y \tag{4}$$

and

$$SUR: \quad \tilde{\beta} = \left(X'\tilde{\Omega}^{-1}X\right)^{-1} X'\tilde{\Omega}^{-1}Y \tag{5}$$

where $\tilde{\Omega}$ is the estimated covariance matrix

$$\tilde{\Omega} = (s_{ij}) \otimes I,$$

s_{ij} is the estimator of σ_{ij}. The Zellner's restricted SUR (abbreviated as RSUR) estimator denoted by $\tilde{\beta}_{RSUR}$ uses the estimator

$$s_{ij} = \frac{\hat{u}_i{}'\hat{u}_j}{T-1},$$

where

$$\hat{u}_i = y_i - x_i\hat{\beta}_i, \quad \hat{\beta} = (x_i{}'x_i)^{-1} x_i{}'y_i, \quad i, j = 1,2.$$

The Zellner's unrestricted SUR (abbreviated as USUR) estimator denoted by $\tilde{\beta}_{USUR}$ uses the estimator s^*_{ij}

$$s^*_{ij} = \frac{e_i{}'e_j}{T-n},$$

where

$$e_i = y_i - Z\hat{\gamma}, \quad Z = (x_1 \quad x_2), \quad \hat{\gamma} = (Z'Z)^{-1} Z'y_i, \quad i, j = 1,2, \quad n = 2.$$

Both estimators s_{ij} and s^*_{ij} are consistent estimators of $\sigma_{ij}, i, j = 1,2.$ (see Appendix).

3. Asymptotic distributions of the OLS and SUR estimators

Following Park and Phillips(1988), we derive the asymptotic distributions of the RSUR, USUR and OLS estimators. We define 2-dimensional vectors W_t:

$$W_t = \begin{pmatrix} v_t \\ w_t \end{pmatrix}.$$

The standardized partial sum of W_t converges in distribution as

$$\frac{1}{\sqrt{T\sigma_{ii}}}\sum_{j=1}^{t} W_{[T\cdot r]} \Rightarrow W(r) = \begin{pmatrix} W_1(r) \\ W_2(r) \end{pmatrix},$$
$$(j-1)/T \le r \le j/T, \quad i = 1,2, \quad j = 1,2,\ldots,T$$

where [ ] denotes the integer part of its argument and "⇒" signifies convergence in distribution and $W(r)$ is a 2-dimensional standardized Brownian motion.

It is easy to derive the asymptotic distribution of the OLS estimator. The result is as follows.

Theorem 1 : *The asymptotic distributions of the OLS estimators are given as*

$$T(\hat{\rho} - \rho) \Rightarrow \frac{(1/2)\left(W_1(1)^2 - 1\right)}{\int_0^1 W_1(r)^2 dr}, \tag{6}$$

$$\begin{pmatrix} T^{1/2}(\hat{c} - c) \\ T^{3/2}(\hat{\beta} - \beta) \end{pmatrix} \Rightarrow N\left(0, \sigma_{22}\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}^{-1}\right) \tag{7}$$

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The normalized SUR estimator $\tilde{\beta}$ is written as

$$\Psi_T\left(\tilde{\beta} - \beta\right) = \Psi_T\left(X'\tilde{\Omega}^{-1}X\right)^{-1} X'\tilde{\Omega}^{-1}U \tag{8}$$

where

$$\Psi_T = \begin{bmatrix} T & 0 & 0 \\ 0 & T^{1/2} & 0 \\ 0 & 0 & T^{3/2} \end{bmatrix}.$$

More concretely we can write it as

$$\begin{bmatrix} T(\tilde{\rho}-\rho) \\ T^{1/2}(\tilde{c}-c) \\ T^{3/2}(\tilde{\beta}-\beta) \end{bmatrix}$$

$$= \begin{bmatrix} s^{22}T^{-2}\sum_{t=1}^{T} y_{t-1}^2 & -s^{12}T^{-3/2}\sum_{t=1}^{T} y_{t-1} & -s^{12}T^{-5/2}\sum_{t=1}^{T} ty_{t-1} \\ -s^{21}T^{-3/2}\sum_{t=1}^{T} y_{t-1} & s^{11} & s^{11}/2 \\ -s^{21}T^{-5/2}\sum_{t=1}^{T} ty_{t-1} & s^{11}/2 & s^{11}/3 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} s^{22}T^{-1}\sum_{t=1}^{T} v_t y_{t-1} - s^{12}T^{-1}\sum_{t=1}^{T} w_t y_{t-1} \\ -s^{21}T^{-1/2}\sum_{t=1}^{T} v_t + s^{11}T^{-1/2}\sum_{t=1}^{T} w_t \\ -s^{21}T^{-3/2}\sum_{t=1}^{T} tv_t + s^{11}T^{-3/2}\sum_{t=1}^{T} tw_t \end{bmatrix} \qquad (9)$$

where s^{ij} is (i, j) element of the inverse matrix $(s_{ij})^{-1}, i = 1,2.$

By the standard calculation as in Phillips and Perron(1988) and Phillips and Park(1988), we have

$$\frac{1}{T^2}\sum_{t=1}^{T} y_{t-1}^2 \Rightarrow \sigma_{11}\int_0^1 W_1(r)^2 dr,$$

$$\frac{1}{T\sqrt{T}}\sum_{t=1}^{T} y_{t-1} \Rightarrow \sqrt{\sigma_{11}}\int_0^1 W_1(r)dr,$$

$$\frac{1}{T^2\sqrt{T}}\sum_{t=1}^{T} ty_{t-1} \Rightarrow \sqrt{\sigma_{11}}\int_0^1 rW_1(r)dr,$$

$$\frac{1}{T}\sum_{t=1}^{T} v_t y_{t-1} \Rightarrow (1/2)\sigma_{11}\left(W_1(1)^2 - 1\right),$$

$$\frac{1}{T}\sum_{t=1}^{T} w_t y_{t-1} \Rightarrow \sqrt{\sigma_{11}\sigma_{22}}\int_0^1 W_1(r)dW_2(r),$$

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} v_t \Rightarrow \sqrt{\sigma_{11}}W_1(1), \quad \frac{1}{\sqrt{T}}\sum_{t=1}^{T} w_t \Rightarrow \sqrt{\sigma_{22}}W_2(1),$$

$$\frac{1}{T\sqrt{T}}\sum_{t=1}^{T} tv_t \Rightarrow \sqrt{\sigma_{11}}\left(W_1(1)-\int_0^1 W_1(r)dr\right),$$

$$\frac{1}{T\sqrt{T}}\sum_{t=1}^{T} tw_t \Rightarrow \sqrt{\sigma_{22}}\left(W_2(1)-\int_0^1 W_2(r)dr\right),$$

$$s_{ij}, s_{ij}^{*} \to \sigma_{ij}, \quad i,j=1,2, \quad s^{ij}, s^{ij*} \to \sigma^{ij}, \quad i,j=1,2$$

where "→" signifies convergence in probability. By substitution we have the asymptotic distributions of the RSUR and USUR estimator as follows.

Theorem 2 : *The asymptotic distributions of the RSUR and USUR estimators are given as*

$$\begin{bmatrix} T(\tilde{\rho}-\rho) \\ T^{1/2}(\tilde{c}-c) \\ T^{3/2}(\tilde{\beta}-\beta) \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \sigma^{22}\sigma_{11}\int_0^1 W_1(r)^2 dr & -\sigma^{12}\sqrt{\sigma_{11}}\int_0^1 W_1(r)dr & -\sigma^{12}\sqrt{\sigma_{11}}\int_0^1 rW_1(r)dr \\ -\sigma^{21}\sqrt{\sigma_{11}}\int_0^1 W_1(r)dr & \sigma^{11} & \sigma^{11}/2 \\ -\sigma^{21}\sqrt{\sigma_{11}}\int_0^1 rW_1(r)dr & \sigma^{11}/2 & \sigma^{11}/3 \end{bmatrix}^{-1} \times$$

$$\begin{bmatrix} \sigma^{22}(1/2)\sigma_{11}\left(W_1(1)^2-1\right)-\sigma^{12}\sqrt{\sigma_{11}\sigma_{22}}\int_0^1 W_1(r)dW_2(r) \\ -\sigma^{21}\sqrt{\sigma_{11}}W_1(1)+\sigma^{11}\sqrt{\sigma_{22}}W_2(1) \\ -\sigma^{21}\sqrt{\sigma_{11}}\left(W_1(1)-\int_0^1 W_1(r)dr\right)+\sigma^{11}\sqrt{\sigma_{22}}\left(W_2(1)-\int_0^1 W_2(r)dr\right) \end{bmatrix} \quad (10)$$

■

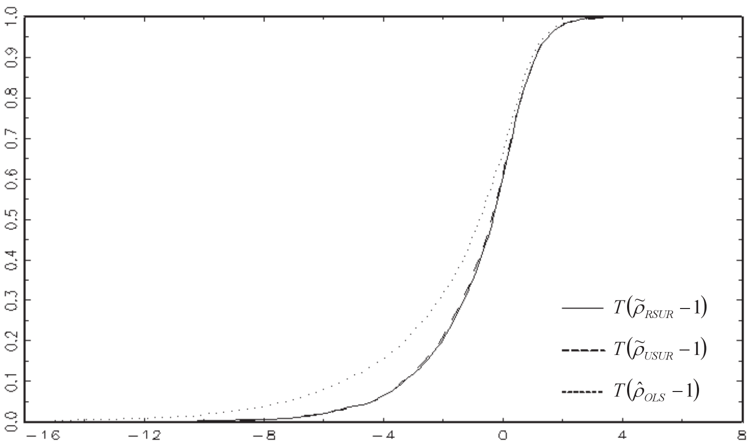
4. Monte Carlo simulations

Small sample performance of $T(\tilde{\rho}_{RSUR}-1)$, $T(\tilde{\rho}_{USUR}-1)$ and $T(\hat{\rho}_{OLS}-1)$ in the 2-equation SUR system are examined by Monte Carlo experiment with 3000 iterations. In what follows, we define $\kappa=\sigma_{22}/\sigma_{11}$ as variance ratio between u_{1t} and u_{2t}, and $r=\sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$ as correlation coefficient between u_{1t} and u_{2t}.

(1) Distribution of the RSUR, USUR and OLS

Figure 4.1 shows a empirical cumulative distribution functions (CDF) of the standardized estimator $T(\tilde{\rho}_{RSUR}-1)$, $T(\tilde{\rho}_{USUR}-1)$ and $T(\hat{\rho}_{OLS}-1)$ when $T=30$, $\rho=1.0$, $r=0.8$, $\kappa=1.0$, $c=1.0$, $\beta=0.5$.

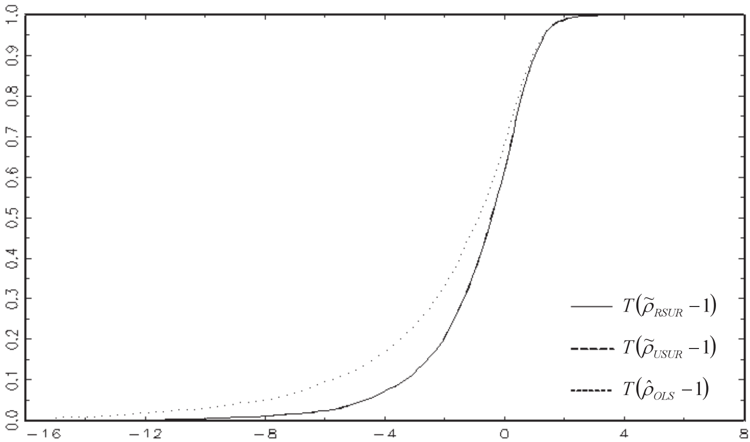
Figure 4.1: CDF of the RSUR, USUR and OLS



Note: $T=30, r=0.8, \kappa=1.0, \rho=1.0, c=1.0, \beta=0.5$

Figure 4.2 is the case for $T=100$, $\rho=1.0$ and $r=0.8$, $\kappa=1.0$, $c=1.0$, $\beta=0.5$.

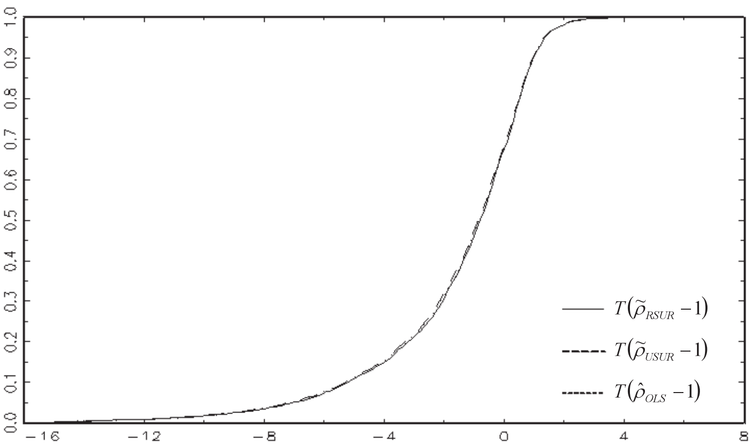
Figure 4.2: CDF of the RSUR, USUR and OLS



Note: $T=100, r=0.8, \kappa=1.0, \rho=1.0, c=1.0, \beta=0.5$

Figure 4.3 is the case for $T=30$, $\rho=1.0$ and $r=0.2$, $\kappa=1.0$, $c=1.0$, $\beta=0.5$.

Figure 4.3: CDF of the RSUR, USUR and OLS



Note: $T=30, r=0.2, \kappa=1.0, \rho=1.0, c=1.0, \beta=0.5$

It is seen that the distributions of the RSUR and USUR estimators are almost indistinguishable but the distributions of the two SUR estimators are more concentrated around the origin than the OLS estimator. The three distributions are asymmetric around the origin.

From them we recognize that the RSUR and USUR estimators are more efficient than the OLS estimator. In Figure 4.3, it is shown that if r is nearly 0, the distributions of these three estimators are indistinguishable.

(2) Power of the RSUR, USUR and OLS for testing $H_0 : \rho = 1$, $H_a : \rho < 1$

We examine the performance of $T(\tilde{\rho}_{RSUR} - 1)$, $T(\tilde{\rho}_{USUR} - 1)$ and $T(\hat{\rho}_{OLS} - 1)$ as a unit root test statistics. We calculate the 5 % critical values of these tests by Monte Carlo experiments at 3000 iterations.

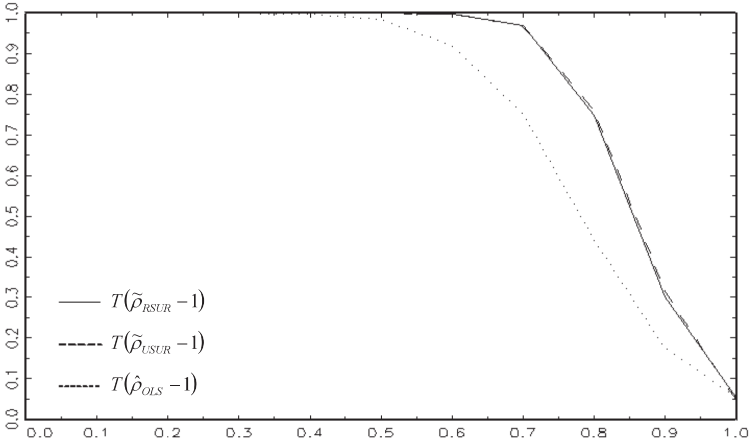
The 5% critical values are −4.48, −4.52, −7.22 for $T = 30$, $\rho = 1.0$, $r = 0.8$, $\kappa = 1.0$, $c = 1.0$, $\beta = 0.5$ and −4.66, −4.65, −8.05 for $T = 100$, $\rho = 1.0$, $r = 0.8$, $\kappa = 1.0$, $c = 1.0$, $\beta = 0.5$. and −7.58, −7.65, −7.66 for $T = 30$, $\rho = 1.0$, $r = 0.2$, $\kappa = 1.0$, $c = 1.0$, $\beta = 0.5$ respectively.

Using these critical points we calculated the power curves of these tests at 3000 replications.

The results are given in Figures 4.4, 4.5, 4.6 respectively. It is shown that the RSUR and USUR estimators are more powerful than the OLS estimator. In Figure 4.6, if r is nearly 0, the power of these test statistics are almost equivalent.

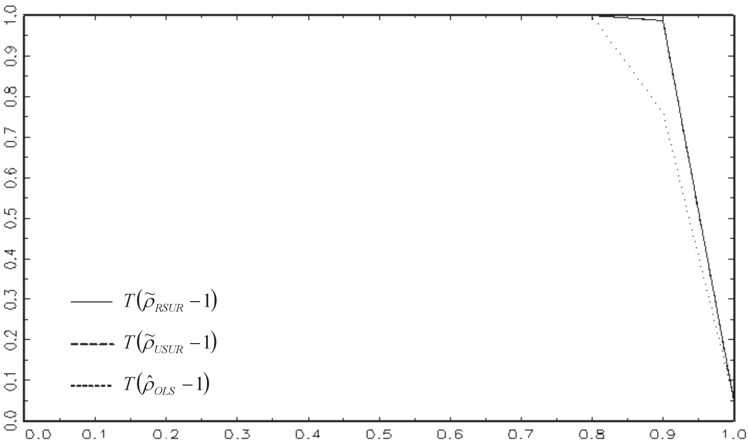
In the above (1) and (2), we also examined the case for $\kappa = 2.0$, but the results were no different from these Figures. Thus they were omitted.

Figure 4.4: Power of the RSUR, USUR and OLS



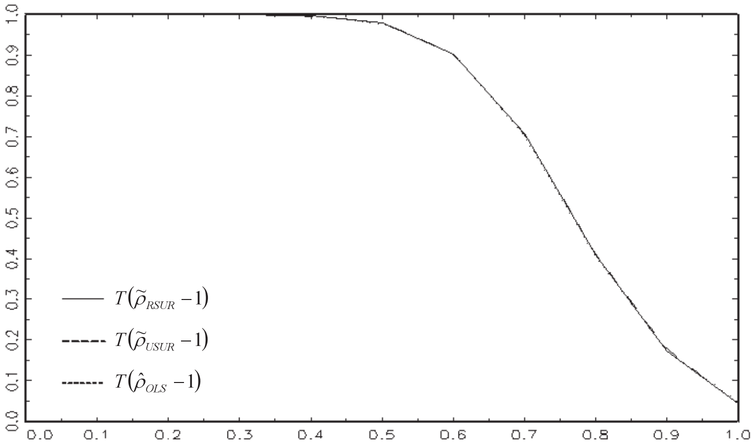
Note: $T = 30, r = 0.8, \kappa = 1.0, \rho = 1.0, c = 1.0, \beta = 0.5$

Figure 4.5: Power of the RSUR, USUR and OLS



Note: $T = 100, r = 0.8, \kappa = 1.0, \rho = 1.0, c = 1.0, \beta = 0.5$

Figure 4.6: Power of the RSUR, USUR and OLS



Note: $T=30, r=0.2, \kappa=1.0, \rho=1.0, c=1.0, \beta=0.5$

5. Concluding remarks

We have considered the Zellner's SUR model in which a random walk and a deterministic trend are mixed with contemporaneously correlated iid disturbances. In the model we derive the asymptotic distributions of the restricted SUR(RSUR), unrestricted SUR(USUR) and OLS estimators. The three estimators of the random walk coefficient ρ have nonstandard asymptotic distributions.

We also analyzed the small sample performance of these estimators. When the correlation coefficient of the error terms is nearly 1, the two SUR estimators are more efficient than the OLS estimator. In this case it is also shown that the two SUR estimators are more powerful than the OLS estimator as a unit root test statistics.

Acknowledgments

We are grateful to Koichi Maekawa for helpful comments.

Appendix

Proof of the $s_{ii}, s_{ii}^* \to \sigma_{ii}$ and $s_{ij}, s_{ij}^* \to \sigma_{ij}, i, j = 1,2.$

(1) case of the RSUR

By the definition of s_{ii}, we have

$$
\begin{aligned}
s_{ii} &= \frac{\hat{u}_i'\hat{u}_i}{T-1} = \frac{y_i'[I - x_i(x_i'x_i)^{-1}x_i']y_i}{T-1} \\
&= \frac{1}{T-1}u_i'u_i - \frac{1}{T-1}\left[\frac{u_i'x_i}{T}\left(\frac{x_i'x_i}{T^2}\right)^{-1}\frac{x_i'u_i}{T}\right] \\
&= \sigma_{ii} + O_p\left(\frac{1}{\sqrt{T}}\right), \quad i = 1,2,
\end{aligned}
$$

where the stochastic order of $u_i'x_i/T$ and $x_i'x_i/T^2$ are $O_p(1)$.

Similarly we have

$$
\begin{aligned}
s_{ij} &= \frac{\hat{u}_i'\hat{u}_j}{T-1} = \frac{y_i'[I - x_i(x_i'x_i)^{-1}x_i'][I - x_j(x_j'x_j)^{-1}x_j']y_j}{T-1} \\
&= \sigma_{ij} + O_p\left(\frac{1}{\sqrt{T}}\right), \quad \forall i \neq j \quad i, j = 1,2.
\end{aligned}
$$

(2) case of the USUR

By the definition of s_{ii}^* we have

$$s_{ii}^* = \frac{e_i'e_i}{T-n} = \frac{y_i'[I - Z(Z'Z)^{-1}Z']y_i}{T-n}$$

$$= \frac{1}{T-n}u_i'u_i - \frac{1}{T-n}u_i'(y_{-1} \quad \mathbf{i} \quad \mathbf{t})\left\{\begin{pmatrix} y_{-1}' \\ \mathbf{i}' \\ \mathbf{t}' \end{pmatrix}(y_{-1} \quad \mathbf{i} \quad \mathbf{t})\right\}^{-1}\begin{pmatrix} y_{-1}' \\ \mathbf{i}' \\ \mathbf{t}' \end{pmatrix}u_i$$

$$= \sigma_{ii} + O_p\left(\frac{1}{\sqrt{T}}\right), \quad i = 1,2, \quad n = 2,$$

where $i' = (1,1,1,\ldots,1)'$ with T elements and $t' = (1,2,3,\ldots,T)'$ is a time trend.

Similarly we have

$$s_{ij}^* = \frac{e_i'e_j}{T-n} = \frac{y_i'[I - Z(Z'Z)^{-1}Z']y_j}{T-n}$$

$$= \frac{1}{T-n}u_i'u_j - \frac{1}{T-n}u_i'(y_{-1} \quad \mathbf{i} \quad \mathbf{t})\left\{\begin{pmatrix} y_{-1}' \\ \mathbf{i}' \\ \mathbf{t}' \end{pmatrix}(y_{-1} \quad \mathbf{i} \quad \mathbf{t})\right\}^{-1}\begin{pmatrix} y_{-1}' \\ \mathbf{i}' \\ \mathbf{t}' \end{pmatrix}u_j$$

$$= \sigma_{ij} + O_p\left(\frac{1}{\sqrt{T}}\right), \quad \forall i \neq j \quad i,j = 1,2, \quad n = 2.$$

■

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