Arrow's Impossibility Theorem and ways out of the impossibility

テキスト

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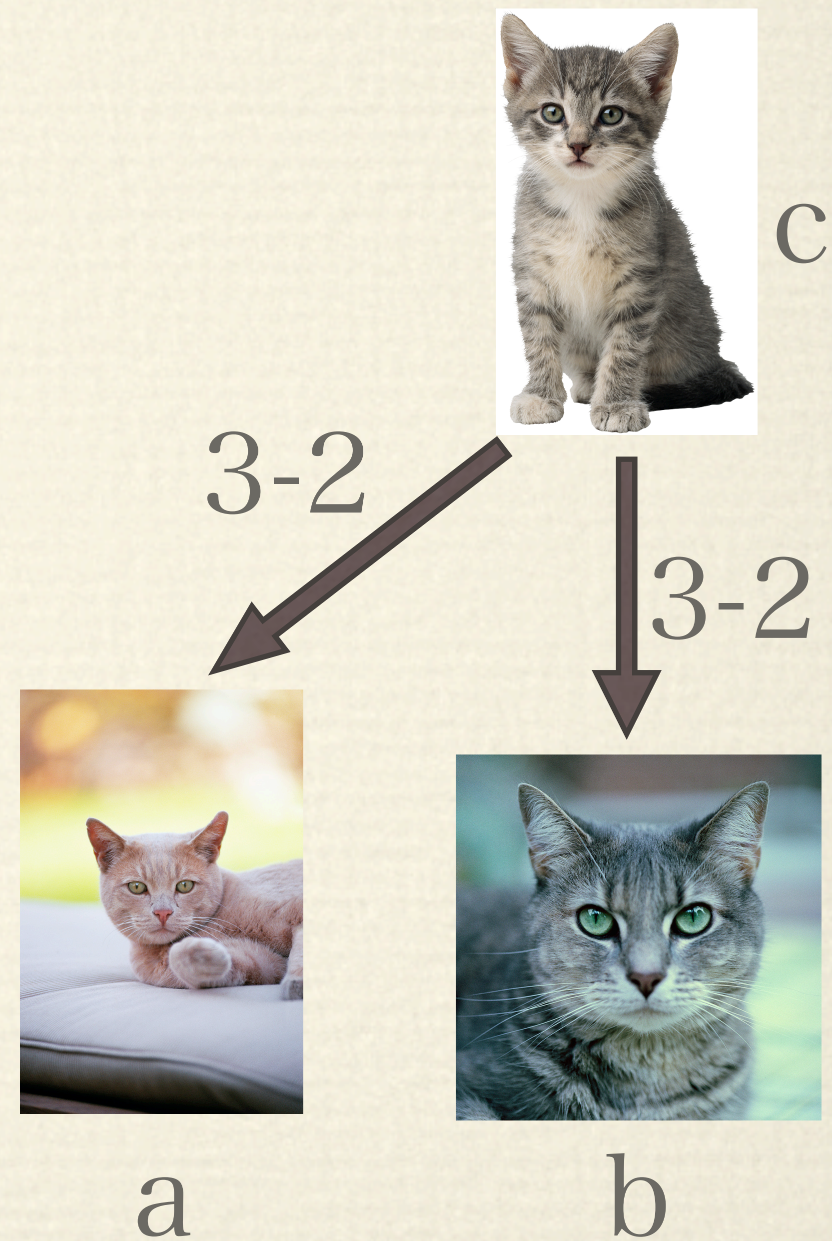
Outline

- Examples
- Impossibility theorem
- Ways out of the impossibility

A Beauty Contest

- Individual rankings (preferences)
  - 3 judges: cba
  - 2 judges: bac
- Plurality rule elects c.
- Condorcet's pairwise comparison also chooses c (**majority winner**):
  - c beats both a and b by a majority of 3 to 2.
- Looks like c is the "right" choice...



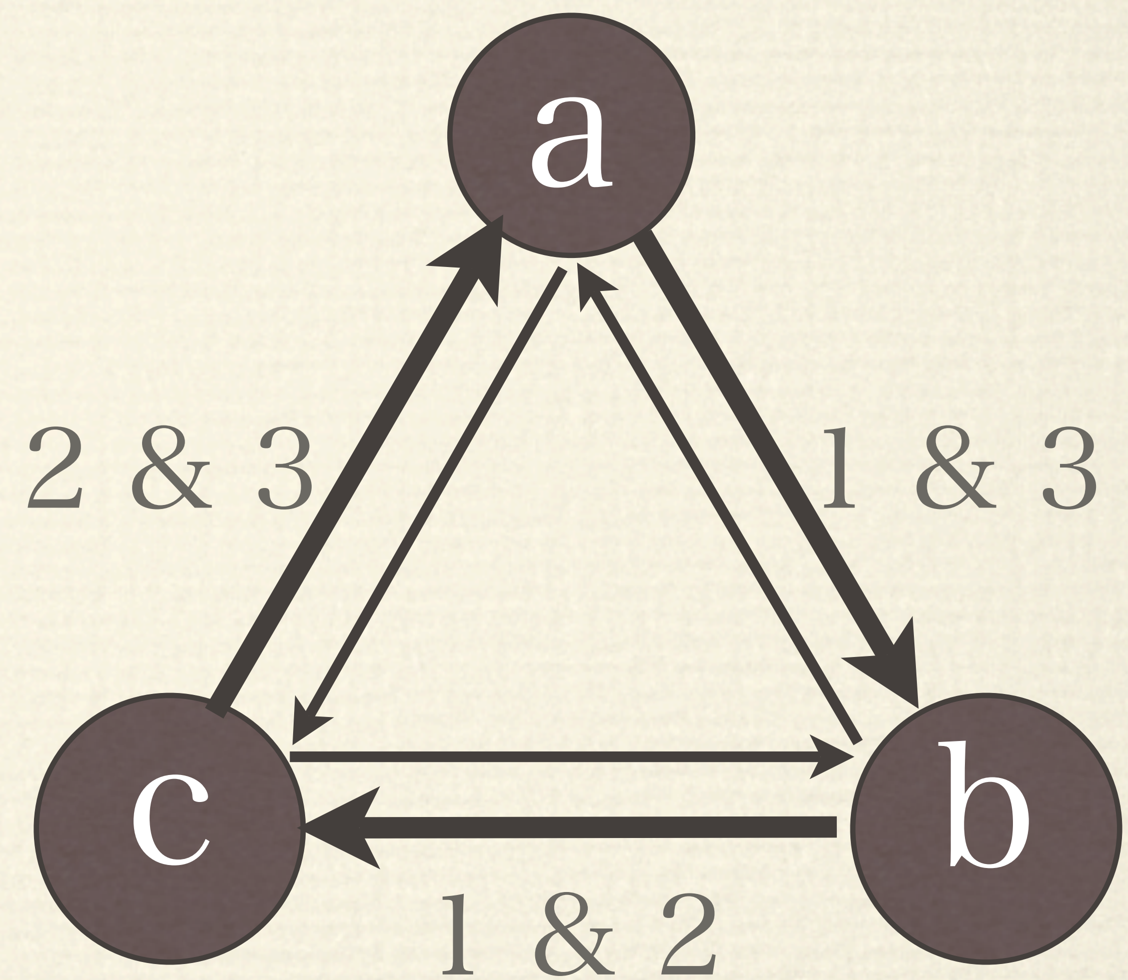
The Borda Rule

3 judges: cba
2 judges: bac

- Each voter (judge) gives
  - 2 points to the 1st alternative in his preference,
  - 1 point to the 2nd,
  - 0 point to the 3rd.
- Total scores:
  - a gets 3*0+2*1=2 pts.
  - b gets 3*1+2*2=7 pts.
  - c gets 3*2+2*0=6 pts.
- The Borda ranking: bca.
- b is the Borda winner.
  - But a majority prefer c to b.

The paradox of Voting

- 3 voters' preferences:
  - Voter 1: abc (aP_1b, bP_1c, aP_1c)
  - Voter 2: bca
  - Voter 3: cab
- Majority preferences form a **cycle.**
  - No maximal ("best") alternative.



An aggregation rule

- For the moment, suppose there are 3 alternatives and 3 voters.
- A **(preference) aggregation rule** is a method for aggregating individual rankings into a single consensus ranking.

profile (R_1, R_2, R_3) of preferences R_i → aggregation rule → **group preference** R

The case of 3 alternatives and 3 voters

- Each voter has 3!=6 possible preferences R_i:
  - abc, acb, bac, bca, cab, cba.
  - (Okay to allow preferences such as [ab]c, a[bc], [abc]. 7 more possibilities.)
- So, there are 6^3=216 inputs (profiles).
- An aggregation rule must specify a preference R for each of the 216 profiles (R_1, R_2, R_3).
  - R can be any of 6+7=13 preferences, because disallowing ties is too restrictive.
- There are many (13^{216}) aggregation rules, including terrible ones.

Arrow's Theorem

- Assume there are **at least 3** alternatives and 2 voters.
- Arrow (1951). There is **no** aggregation rule that satisfies the three conditions:
  - **Unanimity**. If every voter prefers x to y, then the group must rank x above y.
  - **(Pairwise) Independence**. Whether the group ranks x above y depends only on voters' preferences between x and y.
  - **Nondictatorship**. There is no voter whose preference always determines the group preference.

How about the rules we mentioned?

- Pairwise majority voting
  - satisfies Independence, Unanimity, and Nondictatorship;
  - is not an agregation rule.
    - The voting paradox gives a **cyclic group preference**, not one of the 13 rational preferences.
- The Borda rule
  - is an aggregation rule, satisfying Unanimity and Nondictatorship;
  - **violates Independence** (next slide).

The Borda rule violates Independence

- Before:
  - 3 judges: **cba**
  - 2 judges: bac
  - Borda rank: bca
- After:
  - 3 judges: **cab**
  - 2 judges: bac
  - Borda rank: cab

- The group ranked b above c before.
- Individual preferences between b and c is the same as before.
- If Independence is satisfied, the group should rank b above c after the change.
  - But it doesn't.

Ways out of Arrow's impossibility

1. **Infinitely** many voters
    - There are rules satisfying Arrow's conditions (Fishburn, 1970).
    - Mihara (1997 ET; 1999 JME; 2004 MSS) reinterprets "individuals" and considers computational issues.
2. Group choice instead of group preference
    - Nondictatorial functions are manipulable (Gibbard 1973; Satterthwaite 1975).
    - Mihara (2000 SCW; 2001 SCW) considers **group** manipulation.

3. Restricting profiles of preferences
    - Single-peaked preference: Black's Medial Voter Theorem in one dimension (1958).
    - McKelvey's Chaos Theorem (1976) in higher dimensions.

4. Relaxing rationality of group preference
    - Assuming **acyclic** (not cyclic) preferences is enough for maximization.
    - A "simple" aggregation rule is acyclic iff the number of alternatives is less than the **Nakamura number** (Nakamura, 1979).
    - Kumabe and Mihara (2008 JME; 2008 SCW) extend Nakamura's theorem and obtain conditions for a large Nakamura number.
5. Restricting the number of altenratives to 2
    - Only simple majority rule satisfies **anonymity**, **neutrality**, and monotonicity (May, 1952).
    - Mihara (1997 SCW; 2004 MSS) considers anonymity and neutrality without restricting the number of alternatives.

References

- Papers by H.R. Mihara
  http://econpapers.repec.org/RAS/pmi193.htm
- H. Reiju Mihara's website
  http://www5.atwiki.jp/reiju/