COMPACTNESS THEOREMS ON A REAL PROBABILITY SPACE

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SUMMARY

A generalization of Helly's selection theorem to multi-dimensional case [6] states that any sequence of k-dimensional real random variables $\{X_n\}$ $(n=1,2,\ldots)$ contains a subsequence $\{X_n\}(\{m\}\subseteq\{n\})$ which converges in the wide sense.

In this article, via expository measure theoretic approach, it is shown that under certain conditions there exists a subsequence which converges *completely* to some real random variable in the sense of certain types of convergence.

1. INTRODUCTION AND PRELIMINARIES

Let (Ω,F,P) be a basic probability space, and (R,B,P^X) designate an induced probability space by a k-dimensional real random variable X , where R being the k-dimensional Euclidean space and B the usual Borel field of subsets of R. Throughout this article k is assumed to be *fixed* independently of n. Further, let $\mathsf{F}(R,\mathrm{B})$ be the family of all probability distributions over the measurable space (R,B), and $\mathsf{P}(R,\mathrm{B},\mu)$ be its subfamily composed by the all absolutely continuous probability distributions with respect to the usual Euclid-Lebesgue measure over (R,B). In the subsequent discussions we shall denote the members of each family by the random variables X, Y, ... instead of their corresponding probability measures P^X, P^Y, ..., respectively.

Let $\{X_n\}$ $(n=1,2,\ldots)$ and $\{Y_n\}$ $(n=1,2,\ldots)$ be two sequences of k-dimensional random variables belonging to $\mathsf{F}(R,\mathrm{B})$, and C be any given non-empty subclass of B.

According to Ikeda [2] let us consider the following types of the asymptotic equivalence or those of convergence for real probability distributions;

(1°) any two sequences $\{X_n\}$ and $\{Y_n\}$ are said to be *asymptotically equivalent in the sense of type* $((\mathsf{C}))_d$ and denoted by $X_n \sim Y_n$ $((\mathsf{C}))_d$ $(n \to \infty)$, if for every subset E belonging to C, it holds that

$$(1.1) \qquad \left| P^{X_n}(E) - P^{X_n}(E) \right| \to 0 \quad (n \to \infty),$$

(2°) any two sequences $\{X_n\}$ and $\{Y_n\}$ are said to be *asymptotically equivalent in the sense of type* $(\mathsf{C})_d$ and denoted by $X_n \sim Y_n$ $(\mathsf{C})_d$ $(n \to \infty)$, if it holds that

$$(1.2) \qquad \sup\{\left| P^{X_n}(E) - P^{Y_n}(E) \right| : E \in \mathsf{C}\} \to 0, \quad (n \to \infty).$$

Especially, when Y_n's are all identical with some fixed distribution Y belonging to $F(R,\mathsf{B})$, we have the corresponding notions of the convergence to the definitions above;

(1) the sequence $\{X_n\}$ is said to converge *in the sense of type* $((\mathsf{C}))_d$ and denoted by $X_n \to Y$ $((\mathsf{C}))_d$ $(n \to \infty)$, if it holds that $X_n \sim Y_n$ $((\mathsf{C}))_d$, and

(2) the sequence $\{X_n\}$ is said to converge *in the sense of type* $(\mathsf{C})_d$ and denoted by $X_n \to Y$ $(\mathsf{C})_d$ $(n \to \infty)$, if it holds that $X_n \sim Y_n (\mathsf{C})_d$.

To describe the classical compactness theorem in terms of the probability measures, we introduce another weaker convergence than the above ones, which is the generalized one of *convergence in the wide sense* defined in Tucker's book [6]. Let Q^Z be a measure over (R,B) induced by some fixed measurable function Z defined on Ω to R, and suppose that $0 \le Q^Z(E) \le 1$ for all $E \in \mathsf{C}$. We then define that

(3) the sequence $\{X_n\}$ is said to converge *in the sense of type* $[[\mathsf{C}]]_d$ and denoted by $X_n \to Z$ $[[\mathsf{C}]]_d$ $(n \to \infty)$, if for every subset E belonging to C,

(1.3) $$\left|P^{X_n}(E) - Q^Z(E)\right| \to 0, \quad (n \to \infty).$$

It should be remarked that Q^Z is not necessarily a probability measure over $(R,\mathbf{B})$ and hence Z need not be a member of $F(R,\mathbf{B})$. The following result shows this unfortunate difficulties; Let $\{A_n\}$ $(n=1,2,...)$ be a monotone non-decreasing sequence of subsets such that $A_n \in \mathbf{C}$ for each n and $A_n \uparrow R$ as $n \to \infty$. Consider the probability distribution defined over $(R,\mathbf{C})$ given by

$$P^{X_n}(E) = \begin{cases} 0, & \text{for } E \subseteq A_n, \\ 1, & \text{for } E \subset R - A_n, \end{cases}$$

for each n. Then $Q^Z = \lim_{n\to\infty} P^{X_n} = 0$ (the zero measure) for all $E \in \mathbf{C}$, and hence $\{X_n\}$ converges to a certain measurable function Z defined Ω to R. This example illustrates the existence of escaping the probability mass to infinity, although it is an extreme and not a practical case.

Here, let us consider some familiar classes as the basic class $\mathbf{C}$. Let $\mathbf{M}$ be the clss of all k-dimensional infinite intervals which are right-opened, $\mathbf{S}$ the class of all k-dimensional left-closed and right-opened intervals, and $\mathbf{A}$ be the class of all finite disjoint unions of the members of $\mathbf{S}$.

Under these setup we review the historical results to make ours clear in the subsequent sections.

THEOREM 1.1. (Helly's Selection Theorem). *Let* $\{X_n\}$ $(n=1,2,...)$ *be any sequence of* k*-dimensional random variables belonging to* $F(R,\mathbf{B})$. *Then there exist a subsequence* $\{m\} \subseteq \{n\}$ *and a measure* Q^Z *over* $(R,\mathbf{B})$ *induced by some fixed measurable function* Z *defined on* Ω *to* R, *satisfying the condition* $0 \le Q^Z(E) \le 1$ *for all* $E \in \mathbf{M}$, *such that*

(1.4) $$X_m \to Z \quad [[\mathbf{M}]]_d \quad (n \to \infty).$$

This theorem assures us that ths family $F(R,B)$ is *relatively compact* in the sense of type $[[M]]_d$, and is also called a *weak compactness theorem.* Under a certain condition K. S. Rao and D. G. Kendall [5] extended the theorem to a *complete compactness* case, which can be recast in our fashion as follows;

THEOREM 1.2. *Let* $\{X_n\}$ $(n=1,2,\ldots)$ *be any sequence of random variables belonging to* $F(R,B)$ *, and* $\mu_{2,n}$ *be the second moment of* $\{X_n\}$ *satisfying the following condition*

$$(1.5)\qquad \mu_{2,n} = \int x^2\, dP^{X_n} < K < \infty, \qquad \textit{for all} \quad n \geq N,$$

where K *is a positive constant and* N *is an integer. Then there exist a subsequence* $\{m\} \subseteq \{n\}$ *and a* probability measure P^Y *over* (R,B) *induced by some fixed* real random variable Y, *such that*

$$(1.6)\qquad X_m \rightarrow Y \quad ((M))_d \quad (n\to\infty).$$

Under a certain condition more general result holds. In order to this we need a certain kind of uniform stochastic boudedness property of the sequence of random variables. The sequence $\{X_n\}$ $(n=1,2,\ldots)$ is said to have *property* $B(S)$, provided that for any given $\varepsilon > 0$, there exist a bounded subset $B = B(\varepsilon)$ belonging to the class S and a positive integer $N = N(\varepsilon,B)$ such that

$$\inf\left\{ P^{X_n}(B) \,\middle|\, n \geq N \right\} > 1-\varepsilon.$$

(The terminology 'property $B(B)$' was introduced by [2], and some authors call the family of the sequence satisfying this property being '*tight*', if B is a compact set. See Billingsley [1]).

The most elaborate result on compactness theorem for a family of real probability distributions is the following, which being a special case of the well known Prohorov theorem for a family of probability measures on $(S,\mathcal{S})$, here S is a (separable and complete) metric space and S the class of Borel sets in S.

THEOREM 1.3. *Let* $\{X_n\}$ $(n=1,2,\ldots)$ *be any sequence of random variables belonging to* $\mathsf{F}(R,\mathsf{B})$. *Then* $\{X_n\}$ $(n=1,2,\ldots)$ *is* relatively compact, *that is* $\{X_n\}$ $(n=1,2,\ldots)$ *contains a* weakly convergent *subsequence, if and only if it has property* $B(\mathsf{S})$.

Next, we introduce a kind of uniform absolute continuity property of the sequence of random variables. The sequence $\{X_n\}$ $(n=1,2,\ldots)$ is said to have *property* $C(\mathsf{C})$, if for any given $\varepsilon > 0$, there exist a positive number $\delta = \delta(\varepsilon)$ and a positive integer $N = N(\varepsilon,\delta)$, such that

$$\sup\left\{ P^{X_n}(E) \;\middle|\; \mu(E) < \delta,\ E \in \mathsf{C},\ n \geq N \right\} < \varepsilon.$$

This notion is necessary for the underlying convergence to be uniform one which will be discussed in the later section.

The purpose of this article is to generalize the above results to the case of complete compactness in the sense of some strictly stroger convrgence than the weak convergence. In Section 2, some lemmas on asymptotic equivalence of real probability distributions are stated. In Section 3, extensions of foregoing results are given.

2. NECESSARY LEMMAS

Let S^* be the class of all subsets of R of the form

$$E^* = \left\{x = (x_1,\ldots,x_k) \;\middle|\; \begin{array}{l} \alpha_i \leq x_i \leq \beta_i\ ,\ i=1,2,\ldots,k \\ \alpha_i\ ,\ \beta_i\ :\ \text{extended rational} \end{array} \right\}$$

and let A^* be the class of all finite disjoint unions of the members of S^*, that is,

$$A^* = \left\{ \sum_{i=1}^{N} E_i^* \;\middle|\; E_i^* \in S^*,\ i=1,2,\dots \right\}.$$

First, we shall prove the following

LEMMA 2.1. *If one of the sequences* $\{X_n\}$ $(n=1,2,\dots)$ *and* $\{Y_n\}$ $(n=1,2,\dots)$ *has property* $B(S)$ *, then it holds that*

$$((S))_d \rightleftarrows ((S^*))_d . \tag{2.1}$$

PROOF. Since $S^* \subset S$, it is readily seen that $((S))_d \longrightarrow ((S^*))_d$. Therefore, it suffices to show that $((S^*))_d \longrightarrow ((S))_d$.

Assume that $X_n \sim Y_n$ $((S^*))$ $(n \to \infty)$, then it is easy to see that both the sequences have the property $B(S)$. Hence, for any given $\varepsilon > 0$, there exist a positive integer n_0 and a set B belonging to S, whose closure being compact, such that

$$P^{X_n}(B) > 1 - \varepsilon \quad \text{and} \quad P^{Y_n}(B) > 1 - \varepsilon ,$$

for all $n \geq n_0$. For any set S belonging to S, the set E defined by $E = S \cap B$ is bounded, and it holds that

$$\left|P^{X_n}(S) - P^{X_n}(E)\right| < \varepsilon \quad \text{and} \quad \left|P^{Y_n}(S) - P^{Y_n}(E)\right| < \varepsilon ,$$

for all $n \geq n_0$. Note that E belongs to S, then for all $n \geq n_0$,

$$\left|P^{X_n}(S) - P^{Y_n}(S)\right| < \left|P^{X_n}(E) - P^{Y_n}(E)\right| + 2\varepsilon.$$

Since for any set E belonging to S there exists a monotone decreasing sequence of sets in S^*, say $\{E_n^*\}$ $(n=1,2,\dots)$, such that $E_n^* \subseteq B$ for each n and that $\bigcap_{n=1}^{\infty} E_n^* = E$, then there exists a positive integer m satisfying the following inequalities

$$\left|P^{X_n}(E) - P^{X_n}(E_m^*)\right| < \varepsilon \quad \text{and} \quad \left|P^{Y_n}(E) - P^{Y_n}(E_m^*)\right| < \varepsilon,$$

for all $n \geq m$, where ε is the same value as above. Thus, putting $l = \max(n_0, m)$, we have

$$\left|P^{X_n}(S) - P^{Y_n}(S)\right| < \left|P^{X_n}(E_l^*) - P^{Y_n}(E_l^*)\right| + 4\varepsilon,$$

which implies that

$$\left|P^{X_n}(S) - P^{Y_n}(S)\right| \longrightarrow 0, \quad (n \to \infty).$$

Hence, it holds that $X_n \sim Y_n$ $((S))_d$ as $n \to \infty$, which completes the proof.

In the similar manner to the above we can show next results, whose proof will be omitted.

LEMMA 2.2. *If one of the sequences* $\{X_n\}$ $(n=1,2,\ldots)$ *and* $\{Y_n\}$ $(n=1,2,\ldots)$ *has property* $B(S)$, *then it holds that*

$$((A))_d \rightleftarrows ((A^*))_d . \tag{2.2}$$

Since $((A))_d \rightleftarrows ((S))_d$ always holds as was shown in [2], we thus obtain the following

COROLLARY 2.1. *If one of the sequences* $\{X_n\}$ $(n=1,2,\ldots)$ *and* $\{Y_n\}$ $(n=1,2,\ldots)$ *has property* $B(S)$, *then it holds that*

$$((A^*))_d \rightleftarrows ((A))_d \rightleftarrows ((S))_d \rightleftarrows ((S^*))_d . \tag{2.3}$$

LEMMA 2.3. *If at least one of the sequences* $\{X_n\}$ $(n=1,2,\dots)$ *and* $\{Y_n\}$ $(n=1,2,\dots)$ *has property* $B(\mathrm{S})$ *and* $C(\mathrm{S})$ *simultaneously or one for each, then it holds that*

$$((\mathrm{A}))_d \rightleftarrows (\mathrm{M})_d \quad . \tag{2.4}$$

This lemma is directly obtained by Theorem 3.2 in [2] and the fact that $((\mathrm{A}))_d \rightleftarrows ((\mathrm{S}))_d \rightleftarrows ((\mathrm{M}))_d$ holds without any condition.

3. EXTENSIONS OF WEAK COMPACTNESS THEOREMS

In the first place, we consider an extension of Helly's selection theorem to the case of $((\mathrm{A}^*))_d$ convergence.

LEMMA 3.1. *If a sequence of k-dimensional random variables* $\{X_n\}$ $(n=1,2,\dots)$ *has property* $B(\mathrm{S})$, *then there exist a subsequence* $\{m\} \subseteq \{n\}$ *and a real random variable* Y *belonging to* $F(R,\mathrm{B})$, *the class of all 'proper' random variables defined over* (R,B), *such that*

$$X_m \longrightarrow Y \quad ((\mathrm{A}^*))_d \quad (n \to \infty). \tag{3.1}$$

PROOF. Since $0 \le P^{X_n}(E) \le 1$ $(n=1,2,\dots)$ for any subset $E \in \mathrm{B}$ and the class S^* consists of countable subsets of R, we can see, by using the so-called diagonal method, that for some subsequence $\{m\} \subseteq \{n\}$ the probability measure $P^{X_n}(E^*)$ converges for every set E^* belonging to S^*.

Let us put

$$\tilde{Q}(E^*) = \lim_{m \to \infty} P^{X_m}(E^*) \tag{3.2}$$

for each E^* in S^*.

We shall now define the set function $Q(A^*)$ over A^* by

(3.3) $$Q(A^*) = \sum_{i=1}^{N} \tilde{Q}(E^*),$$

where $A^* = \sum_{i=1}^{N} E_i^*$, $E_i^* \in S^*$ $(i=1,\ldots,N)$ and $E_i^* \cap E_j^* = \phi$ $(i \neq j)$. Then, this set function satisfies the conditions that $Q(R) = 1$ and $0 \leq Q(A^*) \leq 1$, and it is finitely additive over S^*, as is easily shown.

Now, we shall show that the set functions defined above by (3.2) and (3.3) can be extended to a probability measure over (R, B). To this end it is required to exhibit that (i) the class B is the smallest σ-field containing A^*, and (ii) Q is σ-additive over A^*.

Suppose that B^* be the minimal σ-field containing A^*, then $A^* \subset A \subset B$. If we assume that $B \subset B^*$, then $A^* \subset A \subset B \subset B^*$. This contradicts the fact that B^* is the minimal σ-field containing A^*. Hence, $B^* \subset B$. The reverse inclusion relation can be shown as the following way. Any subset belonging to S has the form of

$$E = \left\{ x = (x_1,\ldots,x_k) \;\middle|\; \begin{array}{l} a_\nu \leq x < b_\nu, \ \nu=1,\ldots,k \\ a_\nu, b_\nu : \text{extended real} \end{array} \right\}$$

$$\equiv [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_k, b_k)$$

and we can choose monotone sequences of rational numbers; say $\{\alpha_\nu^t\}$, $\{\beta_\nu^t\}$ and $\{\gamma_\nu\}$ $(\nu=1,\ldots,k;\ t=1,2,\ldots)$; such that α_ν^t, β_ν^t, γ_ν^t are extended rational numbers, $\alpha_\nu^t \uparrow a_\nu$, $\beta_\nu^t \uparrow b_\nu$, $\gamma_\nu^t \downarrow b_\nu$ as $t \to \infty$, and $\alpha_\nu^t \leq \beta_\nu^t \leq \gamma_\nu^t$ for every ν and t. Putting that, for every $i = 1,2,\ldots,M$,

$$F_i^{*t} = [\alpha_{1i}^t, \beta_{1i}^t) \times [\alpha_{2i}^t, \beta_{2i}^t) \times \cdots \times [\alpha_{ki}^t, \beta_{ki}^t),$$

and

$$G_{\nu i}^{*t} = [\beta_{\nu 1 i}^t, \infty) \times [\beta_{\nu 2 i}^t, \infty) \times \cdots \times [\alpha_{\nu\nu i}^t, \gamma_{\nu\nu i}^t) \times \cdots \times [\beta_{\nu k i}^t, \infty),$$

then

$$\begin{cases} \bigcap_{t=1}^{\infty} F_i^{*t} = [a_{1i}, b_{1i}] \times [a_{2i}, b_{2i}] \times \cdots \times [a_{ki}, b_{ki}] \\ \bigcap_{t=1}^{\infty} G_{\nu i}^{*t} = [b_{\nu 1t}, \infty) \times [b_{\nu 2t}, \infty) \times \cdots \times [a_{\nu\nu i}, b_{\nu\nu i}] \times \cdots \times [b_{\nu\nu i}, \infty). \end{cases}$$

Since an arbitrary subset A in A is represented by $A = \sum_{i=1}^{M} E_i$, where E_i $(i=1,\ldots,M) \in \mathsf{S}$ with $E_i \cap E_j = \phi$ $(i \neq j)$, then for each E_i there exist certain sequences of sets in S^*, say $\{F_i^{*t}\}$ $(t=1,2,\ldots;\ i=1,\ldots,M)$ and $\{G_{\nu i}^{*t}\}$ $(t=1,2,\ldots;\ i=1,\ldots,M;\ \nu=1,\ldots,k)$ respectively, such that

$$E_i = \bigcap_{t=1}^{\infty} F_i^{*t} - \bigcup_{\nu=1}^{k} \bigcap_{t=1}^{\infty} G_{\nu i}^{*t} .$$

Hence $\mathsf{A} \subset \mathsf{B}^*$. Noting that B is the minimal σ-field containing A, we obtain $\mathsf{B} \subset \mathsf{B}^*$. Therefore, B^* coincides with the Borel field B, which proves (i).

We must now show (ii). For every set E belonging to B, let $\overline{Q}$ be an outer measure induced by the measure Q;

$$(3.4)\qquad \overline{Q}(E) = \inf \left\{ \sum_{i=1}^{\infty} Q(A_i^*) \;\middle|\; E \subset \sum_{i=1}^{\infty} A_i^*,\ A_i^* \in \mathsf{A}^*,\ i=1,2,\ldots \right\}.$$

Then, for each i, choosing a covering of $A_i^* \in \mathsf{A}^*$, such that

$$A_i^* \subset \sum_{j=1}^{\infty} A_{ij}^* \ ;\ A_{ij}^* \in \mathsf{A}^* \quad (i=1,2,\ldots),$$

it follows from (3.4) that for any given $\varepsilon \geq 0$

$$\sum_{j=1}^{\infty} Q(A_{ij}^*) \leq \overline{Q}(A_i^*) + \varepsilon / 2^i \quad (i=1,2,\ldots).$$

Here, put $E = \sum_{i=1}^{\infty} A_i^*$, then $E \subset \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij}^*$ and that

$$\overline{Q}(E) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \overline{Q}(A_{ij}^*) \leq \sum_{i=1}^{\infty} Q(A_i^*) + \varepsilon .$$

Since Q is an restriction of $\overline{Q}$ on A^* and $\varepsilon > 0$ is arbitrary, then the above implies

$$(3.5) \qquad \overline{Q}(E) \leq \sum_{i=1}^{\infty} Q(A_i^*).$$

Next, we must prove the contrary inequality of (3.5). Noting that the monotone non-decreasingness of $\overline{Q}(E)$, we get for any positive integer N

$$\overline{Q}(E) = \overline{Q}(\sum_{i=1}^{\infty} A_i^*) \geq \overline{Q}(\sum_{i=1}^{N} A_i^*) \geq \sum_{i=1}^{N} \overline{Q}(A_i^*) = \sum_{i=1}^{N} Q(A_i^*) .$$

The arbitrariness of N implies that

$$(3.6) \qquad \overline{Q}(E) \geq \sum_{i=1}^{\infty} Q(A_i^*) .$$

Hence, from (3.5) and (3.6), it follows that, for $E = \sum_{i=1}^{\infty} A_i^*$ with $A_i^* \in \mathsf{A}^*$, $i=1,2,\dots,$

$$(3.7) \qquad \overline{Q}(E) = \sum_{i=1}^{\infty} Q(A_i^*) .$$

which means that Q is σ-additive over A^*.

Therefore, the extension theorem assures us that Q can be uniquely extended to a probability measure $\overline{Q}$ over B by (3.4). (Cf. Kolmogorov [3]). We shall denote a random variable corresponding to this probability measure by Y. It is clear that Y belonging to $F(R,\mathsf{B})$, because property $B(S)$ of the sequence $\{X_n\}(i=1,2,\dots)$ is brought over Y, too. Putting $\overline{Q} = P^Y$, we can now see that

$$P^{X_m}(A^*) \longrightarrow P^Y(A^*) \quad (n \to \infty),$$

for every set A^* belonging to A^*. This comletes the proof of the lemma.

Now, we are in a position to state the following result:

THEOREM 3.1. *Let* $\{X_n\}$ $(n=1,2,\ldots)$ *be any sequence of k-dimensional rndom variables belonging to the class of all proper random variables defined over* (R,B). *Then, there exist a subsequence* $\{m\} \subseteq \{n\}$ *and a real random variable belonging to* $\mathrm{F}(R,\mathrm{B})$, *such that*

$$X_m \longrightarrow Y \quad ((\mathrm{A}))_{\mathrm{d}} \quad (n \to \infty), \tag{3.8}$$

if and only if it has property $B(\mathrm{S})$.

PROOF. The "if" part immediately follows by Lemmas 3.1 and 2.2. As for the "only if" assertion, suppose that the sequence $\{X_n\}$ $(n=1,2,\ldots)$ does not have property $B(\mathrm{S})$. Then, for some $\varepsilon > 0$, for any bounded subset $B = B(\varepsilon)$ belonging to S and for any positive integer $N = N(\varepsilon, B)$, there exists a positive integer n, such that

$$P^{X_n}(B) \leq 1 - \varepsilon, \quad \text{for all} \quad n \geq N.$$

Define $B_N = \{ x \mid -N \leq x_i < N,\ i=1,\ldots,k \}$ with $N = [1/\varepsilon]$ and put $A_N = E \cap B_N$ for any subset E belonging to A. If there exist a subsequence $\{m\} \subseteq \{n\}$ and a random variable Y belonging to $\mathrm{F}(R,\mathrm{B})$ such that $X_m \to Y$ $((\mathrm{A}))_{\mathrm{d}}$ $(n \to \infty)$, then for every $m \geq N$ it would follow that

$$P^Y(A_N) = \lim_{m\to\infty} P^{X_m}(A_N) \leq \lim_{m\to\infty} P^{X_m}(B_N) \leq 1 - \varepsilon.$$

Especially taking the entire space R as E, it follows that $P^Y(A_N) \longrightarrow P^Y(R) \leq 1 - \varepsilon$ as $N \to \infty$, which contradicts the fact that $Y \in \mathrm{F}(R,\mathrm{B})$, and we complete the proof of the theorem.

Next, we shall state a result on the uniform convergent subsequence whose limiting distribution is absolutely continuous with respect to the Euclid-Lebesgue measure μ.

THEOREM 3.2. *Let* $\{X_n\}$ $(n=1,2,\ldots)$ *be any given sequence of k-dimensional random variables belonging to the class of all proper rmadom variables defined over* (R,B). *Then, there exist a subsequence* $\{m\} \subseteq \{n\}$ *and a real random variable* Y *belonging to* $\mathrm{P}(R,\mathrm{B},\mu)$, *such that*

$$X_m \longrightarrow Y \quad (\mathrm{M})_{\mathrm{d}} \quad (n \to \infty), \tag{3.9}$$

provided that the sequence $\{X_n\}$ $(n=1,2,\ldots)$ *has properties* $B(\mathrm{S})$ *and* $C(\mathrm{S})$ *simultaneously.*

PROOF. Since $\{X_n\}$ $(n=1,2,\ldots)$ has property $B(\mathrm{S})$, by virtue of preceding theorem, there exists a subsequence $\{m\} \subseteq \{n\}$ such that

$$X_m \longrightarrow Y \quad ((\mathrm{A}))_{\mathrm{d}} \quad (n \to \infty), \tag{3.10}$$

where Y is a certain random variable belonging to $\mathrm{F}(R,\mathrm{B})$. From the definition of property $C(\mathrm{S})$ of $\{X_n\}$ $(n=1,2,\ldots)$, the subsequence $\{X_m\}$ has also the property; for any given $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ and a positive integer n_0 such that for every subset A belonging to $\mathbf{A}$,

$$\mu(A) < \delta \tag{3.11}$$

implies that for all $m \geq n_0$

$$P^{X_m}(A) < \varepsilon. \tag{3.12}$$

Thus, it follows that for all $m \geq n_0$

$$P^{Y}(A) \leq \varepsilon + \left|P^{X_m}(A) - P^{Y}(A)\right| .$$

Since, by (3.10), the second term of the RHS in the above tends to zero as $m \to \infty$, then for sufficiently large m

$$P^{Y}(A) < \varepsilon. \tag{3.13}$$

Now, for any subset E in B, let $\{A_i\}$ $(i=1,2,\ldots)$ be a covering of E consisting of mutually disjoint members of A, such that $\mu(A_i) < \delta/2^i$ and $P^Y(A_i) < \varepsilon/2^i$ $(i=1,2,\ldots)$, then by (3.11) it follows that

$$\mu(E) < \mu(\textstyle\sum_i A_i) \leq \delta, \tag{3.14}$$

and this implies that, via (3.13),

$$P^Y(E) < P^Y(\textstyle\sum_i A_i) \leq \varepsilon. \tag{3.15}$$

Therefore, Y belongs to the family $P(R,\mathrm{B},\mu)$, and hence (3.9) follows by Lemma 2.3 and the well-known Polya theorem. Thus, the proof of the theorem is completed.

REMARKS: (i) It is enough supposed that Theorem 3.1 and Theorem 1.3 is closely related. But, out of accordance with our initial purpose, up to now the inclusion relation is not necessarily clear between the relatively compactness and the existence of $((A))_d$- convergent subsequence. In both theorems, anyway, we see that the assumption of propery $B(S)$ is essentially important for the underlying sequence to be relatively compact or completely compact. This property is naturally understood as a generalized notion instead of the conditon $\mu_{2n} < \infty$ assumed in Theorem 1.2.

(ii) Under the condition of Theorem 3.2 $(M)_d \rightleftarrows ((M))_d \rightleftarrows ((G))_d$ holds, where G is the class of all open subsets of R with respect to the usual Euclidean distance. (Cf. [2], [4]). Then, the convergence in (3.9) becomes equivalent to the weak convergence with a proper limiting probability distribution. However, it seems to the author that this theorem is independently interesting and favorable to know the structure of uniform compactness based on the theory of asymptotic equivalence theory for real probability distributions.

(iii) Although we restricted our discussions to the case of fixed k-dimensional real probability spaces, it is of interest to extend our results to more general cases. But, it is still open whether the parallel versions in our notion to the results stated in Billingsley's book [1] are also valid or not.

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