ADJUSTMENT COSTS AND INVESTMENT BEHAVIOUR OF FIRM

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1. Introduction

In this paper, we are interested in considering the investment behaviour of the competitive firm under so-called adjustment costs. So far a number of articles have argued on this theme. The approaches seem to be divided roughly into two groups The first introduces the adjustment costs into a production function, as represented in Lucas ([4]), Gould ([2]) and Treadway ([5], [6]) and others. The second considers the adjustment costs as required in the capital accumulation, as represented in Uzawa ([7] etc.) and others.

We intend to restrict our analysis to the consideration of the following two points: (1) The adjustment costs are shown as a modification of a production function. In addition, a separability assumption is adopted; that is, $Q=F(K,L,\dot{K})=F(K,L)-C(\dot{K})$. This assumption is shown in Treadway [5]. (2) We assume that the time horizon of the firm's investment plan is a finite number T and its growth paths are constrained by $K(T)\geqq K_T$ where K_T denotes the minimum volume of the capital stock required at the terminal point of time.

2. Selected Notations

We begin with making a list of the variables frequehtly seen in this paper.

P : the price of output

W: the wage-rate

G : the price of capital goods

K : the stock of capital goods

C : so-called adjustment costs

L : labour input

r : the rate of discount (>0)

μ : the rate of depreciation (>0)

I : the rate of gross investment

t : a point of time

f, F, H, M, q: some functions to be exactly defined later

For any function $X(\cdot)$, $\dot{X}(t)$ refers to $dX(t)/dt$. Moreover, X' or X_y stands for dX/dy and X'' or X_{yy} for d^2X/dy^2 when X is a function of any variable y.

3. A Model

Our firm is assumed to behave such that he determines his optimum investment by solving the following problem.

Ask the investment level $I(t)$ so as to maximize

$$(1) \qquad V=\int_0^T [PQ(t)-WL(t)-GI(t)]\exp(-rt)dt$$

subject to

$$(2) \qquad Q(t)=F[K(t),L(t)]-C[\dot{K}(t)]$$

$$(3) \qquad \dot{K}(t)=I(t)-\mu K(t)$$

$$(4) \qquad K(0)=K_0(\text{given}),\ K_0>0$$

$$(5) \qquad K(T)\geqq K_T(\text{given}),\ K_T>K_0>0$$

where $Q(t)$ is quantity of the output and T is that time horizon of this firm's investment plan which is taken as some positive constant.

Assumption 1. $F[K(t), L(t)]$ is the linear homogeneous production function with the so-called neoclassical conditions.

Assumption 2. The adjustment cost function $C(\dot{K})$ is continuously twice-differentiable, and satisfies

(6) $C'(\dot{K}) \gtreqless 0$ as $\dot{K} \gtreqless 0$,

(7) $C''(\dot{K}) > 0$ for all $\dot{K}$.

We shall call $C'(\dot{K})$ the marginal adjustment costs.

Assumption 3. The firm is a price taker which means that the prices (P, W, G) are taken as given and he has a stationary expectation on the future prices. That is, (P, W, G) are constant over time.

4. Necessary Conditions

The problem thus formulated may be solved by the well-known Pontryagin's Maximum Principle. Namely, in order that the admissible control $\{I(t), L(t)\}$ and the corresponding trajectory $K(t)$, $0 \leqq t \leqq T$, yield a solution of the optimum problem above, it is necessary that there exists a continuous function $q(t)$, $t \in [0, T]$, such that the following conditions are satisfied:

(a) $$\frac{d[q(t)\exp(-rt)]}{dt} = -\frac{\partial H}{\partial K}$$

(b) $$H[q(t), K(t), L(t), I(t)] = \max_{I(t), L(t)} H[q(t), K(t), L(t), I(t)]$$

(c) $q(T) \geqq 0$, $q(T)[K(T) - K_T] = 0$ (transversality condition)

where the Hamiltonian $H(q, K, L, I)$ is defined as

(8) $$H(q, K, L, I) = \exp(-rt)\{PF[K(t), L(t)] - PC[I(t) - \mu K(t)] - WL(t) - GI(t) + q(t)[I(t) - \mu K(t)]\}.$$

From the condition (a), we get

(9) $\dot{q}(t) = (r+\mu)q(t) - PC'(I - \mu K) - PF_K(K, L)$

Analyzing the condition (b) in connection with (8) in detail, let us first try to maximize H with resprct to $L(t)$. Since it ia easily shown that H is concave to $L(t)$ by $F_{LL}(K, L)<0$, $L(t)$ satisfying the condition (b) is uniquely determined by solving the first-order condition:

$$\frac{\partial H}{\partial L}=\exp(-rt)\left[P\frac{F_L(K,L)}{L}-W\right]=0$$

i.e.,

(10) $F_L(K, L)=\dfrac{W}{P}$.

Then we obtain the following proposition.

Proposition 1

Under the assumptions (1) and (3), the capital-labour ratio $k(t)=K(t)/L(t)$ is uniquely determined and constant over time.

Proof: By Assumption 1, we get $F(K, L)=L(\frac{K}{L}, 1)=Lf(k)$. Therefore, $F_L(K,L)=f(k)-kf\ (k)$. We then have, by(10),

(11) $f(k)-kf'(k)=\dfrac{W}{P}$.

On the other hand, as P and W are given, $k(t)$ is uniquely determined and constant over time by Assumption 3. (Q. E. D.)

Now let us condider the optimum control of H with respect to $I(t)$. To simplify the analysis, we define

(12) $M(I)=-PC(I-\mu K)-GI+qI$.

Then it is easily shown that the following holds

(13) $\max\limits_{I(t)} H \Longleftrightarrow \max\limits_{I(t)} M[I(t)]$.

By using the above result, we get the following proposition.

Proposition 2

The optimum control $I^*(t)$ which satisfies the condition (b) is

$$(14)\qquad I(t)\begin{cases} >0, \text{ if } q(t)>G+PC'(-\mu K(t)) \\ =0, \text{ if } q(t)=G+PC'(-\mu K(t)) \\ <0, \text{ if } q(t)<G+PC'(-\mu K(t)) \end{cases}$$

Proof: Since $M(I)$ is continuous and we have

(15) $$\frac{\partial M(I)}{\partial I} = -PC'(I-\mu K)-G+q \text{ and}$$

(16) $$\frac{\partial^2 M(I)}{\partial I^2} = -PC'(I-\mu K)<0,$$

it follows that $M(I)$ is concave to I. In order to consider the sign of $\partial M(I)/\partial I$ in the equation (15), let us divide (15) into three cases.

Case I. $\left.\frac{\partial M(I)}{\partial I}\right|_{I=0}<0$

In this case, the maximum value of $M(I)$ is obtained when I is negative ($I<0$), as illustrated in Figure 1. Therefore, if $q(t)<G+PC'(-\mu K(t))$, then $I^*(t)<0$.

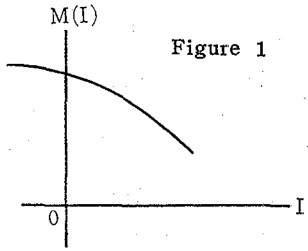

Figure 1

Case II. $\left.\frac{\partial M(I)}{\partial I}\right|_{I=0}=0$

In this case, $M(I)$ is maximized when I is zero, as illustrated in Figure 2. Therefore, if $q(t)=G+PC'(-\mu K(t))$, then $I^*(t)=0$.

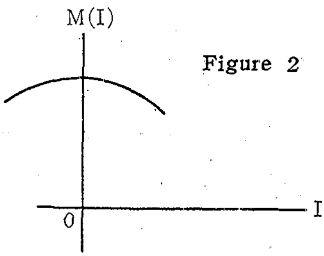

Figure 2

Case III. $\left.\frac{\partial M(I)}{\partial I}\right|_{I=0}>0$

In this case, $M(I)$ is maximized when I is positive, as illustrated in Figure 3. Therefore, if $q(t)>G+PC'(-\mu K(t))$, then $I^*(t)>0$. (Q. E. D.)

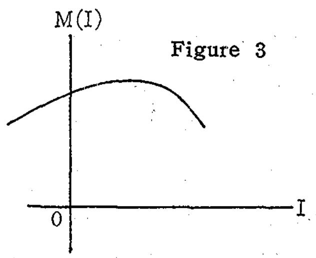

Figure 3

Regarding $q(t)$ and $G+PC'(-\mu K)$ as the demand price of the capital good and the supply price of it respectively, it may be possible to interpret the above criteria economically.

Let us now illustrate the function

(17) $q=G+PC'(-\mu K)$

in the (q, K) plane. By differentiating the equation (17) with respect to K, we obtain

(18) $\dfrac{dq}{dK}=-P\mu C''(-\mu K)<0$

Furthermore, it is easily shown that if $K=0$, then $q=G$ and that if $q=0$, then $K=\hat{K}$ by $C'(-\mu K)=\dfrac{G}{P}\equiv g$. Hence, the equation (17) is illustrated as in Figure 4.

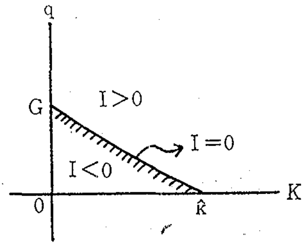

Figure 4

5. The Existence and Uniqueness

In this section, we shall consider the following system of differential equations, (9) and (3):

(9) $\dot{q}(t)=(r+\mu)q(t)-PC'[I(t)-\mu K(t)]-PF_K[K(t),L(t)]$

(3) $\dot{K}(t)=I(t)-\mu K(t)$

First, if $\dot{q}=0$, then it holds that

(19) $q(t)=\dfrac{P}{r+\mu}\{C'[I(t)-\mu K(t)]+F_K[K(t),L(t)]\}$

(20) $\left.\dfrac{\partial q}{\partial K}\right|_{\dot{q}=0}=\dfrac{P}{r+\mu}[C''(I-\mu K)(-\mu)+F_{KK}(K,L)]<0$

Thus, in this case, $q(t)$ is decreasing and typically illustrated as in Figure 5.

Next, if $\dot{K}=0$, then it holds that $K(t)=I(t)/\mu$, but we need to consider the relation between $K(t)$ and $q(t)$. From the familiar Kuhn-Tucker theorem, the condition $\max_I H$ implies the following:

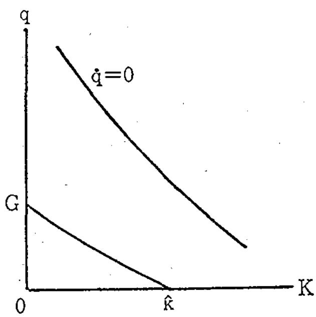

Figure 5

(21) $$\frac{\partial H}{\partial I}=\frac{\partial M(I)}{\partial I}=0.$$

By calculating the equation (21), we obtain

(22) $q(t)=G+PC'[I^*(t)-\mu K(t)]$

Since $\dot{K}=0$, substituting $I^*(t)=\mu K(t)$ into the equation (22), we get

(23) $q(t)=G$

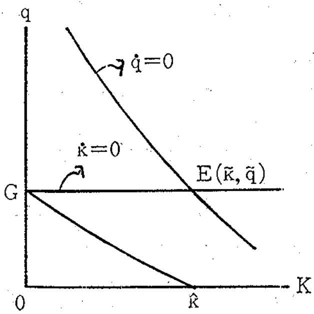

Figure 6

Hence the point of equilibrium E exists uniquely in non-negative phase as illustrated in Figure 6. Then, the following proposition holds.

Proposition 3

At the point of equilibrium E, it holds that

(24) $$F_K(K,L)=\frac{G}{P}(r+\mu).$$

Proof: By solving the system of equations, (19) and (23), we obtain the required result (24). (Q. E. D.)

6. Characterization of The equilibrium solution

Proposition 4

If we define

(25) $$\Psi[K(t),\ q(t)]=\{I(t):S[K(t),\ q(t),\ I(t)]\equiv-G-P\cdot C'(I-\mu K)+q=0,\ I(t)>0,\ K(t)>0,\ q(t)>0\},$$

then it holds

(26) $$\left.\frac{\partial I}{\partial K}\right|_*=\mu$$

(27) $$\left.\frac{\partial I}{\partial q}\right|_*=\frac{1}{PC''(0)}$$

where the asterisk * indicates the evaluation at the point of equilibrium E.

Proof: In order to prove this proposition, we shall apply the Implicit Function Theorem. By the fact that the continuity of $\frac{\partial S}{\partial K}$, $\frac{\partial S}{\partial q}$ and $\frac{\partial S}{\partial I}$ in the neighborhood of the equilibrium point E are satisfied, the three relationships below hold

a) $I(t)=I[K(t), q(t)]$

b) $S\{K(t), q(t), I[K(t), q(t)]\}=0$

c) $I(t)=I[K(t), q(t)]$ is defferentiable at the point $K=\tilde{K}$, $q=\tilde{q}=G$, and

$$\frac{\partial I(K, q)}{\partial q}\bigg|_{*}=-\frac{\partial S/\partial K}{\partial S/\partial I}\bigg|_{*} \tag{28}$$

$$\frac{\partial I(K, q)}{\partial q}\bigg|_{*}=-\frac{\partial S/\partial q}{\partial S/\partial I}\bigg|_{*} \tag{29}$$

Then there exists the only one single-value continuous function for which the conditions (a), (b) and (c) are satisfied. (Implicit Function Theorem) Thus, by considering the following relationships,

$$\frac{\partial S}{\partial K}\bigg|_{*}=-PC''(0)(-\mu)=P\mu C''(0) \tag{30}$$

$$\frac{\partial S}{\partial I}\bigg|_{*}=-PC''(0) \tag{31}$$

$$\frac{\partial S}{\partial q}\bigg|_{*}=1, \tag{32}$$

it holds that

$$\frac{\partial I}{\partial K}\bigg|_{*}=-\frac{P\cdot\mu\cdot C''(0)}{-P\cdot C''(0)}=\mu \tag{33}$$

$$\frac{\partial I}{\partial q}\bigg|_{*}=-\frac{1}{-P\cdot C''(0)}=\frac{1}{P\cdot C''(0)}\ . \tag{34}$$

(Q. E. D.)

We shall now show that the point of equilibrium E is a saddle point.

Proposition 5

The point of equilibrium E is a saddle point.

Proof: In order to proving this Proposition 5, let us consider tnh following characteristic equation.

$$(35)\qquad \left| \begin{matrix} \dfrac{\partial \dot{K}}{\partial K} - \theta & \dfrac{\partial \dot{K}}{\partial q} \\ \\ \dfrac{\partial \dot{q}}{\partial K} & \dfrac{\partial \dot{q}}{\partial q} - \theta \end{matrix} \right|_{*} = 0$$

Then, by taking the following relations account into

$$(36)\qquad \left.\frac{\partial \dot{K}}{\partial K}\right|_{*} = \left.\frac{\partial I}{\partial K}\right|_{*} - \mu = \mu - \mu = 0$$

$$(37)\qquad \left.\frac{\partial \dot{K}}{\partial q}\right|_{*} = \left.\frac{\partial I}{\partial q}\right|_{*} = \frac{1}{P \cdot C''(0)}$$

$$(38)\qquad \left.\frac{\partial \dot{q}}{\partial K}\right|_{*} = P \cdot \mu \cdot C''(0) - PF_{KK}(K,\ L)$$

$$(39)\qquad \left.\frac{\partial \dot{q}}{\partial q}\right|_{*} = r + \mu,$$

we can solve this characteristic equation (38), which can reduced to

$$\theta^2 - (r+\mu)\theta - \frac{\mu \cdot C''(0) - F_{KK}(\tilde{K}, \tilde{L})}{C''(0)} = 0.$$

Thus,

$$(40)\qquad \theta = \frac{1}{2}\{(r+\mu) \pm \sqrt{(r+\mu)^2 + 4(\mu - \frac{F_{KK}(\tilde{K}, \tilde{L})}{C''(0)})}\}.$$

Therefore, (35) has two real roots with opposite signs as shown in (40). (Q. E. D.)

7. The Property of Optimum Paths (A turnpikes theorem)

Since we showed that there are two paths both of which converge to the point of equilibrirm E, let us now illustrate a phase diagram in a $(K,\ q)$ plane. This is shown as in Figure 7.

Thus, we shall prove the following Theorem:

Theorem: a turnpike theorem

For an arbitrary $\varepsilon(>0)$, let us define $N(\varepsilon)$ as follows:

(41) $N(\varepsilon)=\{(K,q): |K-\tilde{K}| \leqq \varepsilon, |q-\tilde{q}| \leqq \varepsilon\}$.

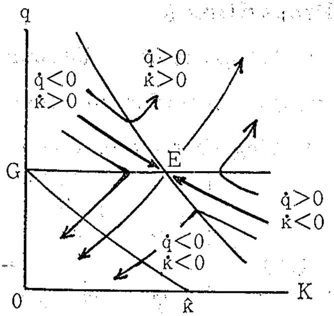

Figure 7

Then, for the unique optimum growth path $\{(K(t), q(t)): 0 \leqq t \leqq T\}$ specified by the initial and terminal parameters (K_0, K_T, T), there exist two finite times $0 \leqq T_1 < \infty$ and $0 \leqq T_2 < \infty$,

$$T_1=T_1(\varepsilon, K_0),\ T_2=T_2(\varepsilon, K_T),$$

such that $\{K(t), q(t)\} \epsilon N(\varepsilon)$ whenever $T_1 \leqq t \leqq T-T_2$.

proof: On proving this theorem, we consider the two cases separately,

(i) the case of $K_0<K_T<\tilde{K}$

(ii) the case of $K_0<\tilde{K}<K_T$.

We also only consider the case for which $K_T<\tilde{K}$, as the alternative case is essentially similar.

Proof of case (i)

The paths which start from a given K_0 and reach $K(T) \geqq K_T$ at a given T are illustrated as in Figure 8.

For an arbitrarily small ε, we shall now make the ε-neighborhood $N(\varepsilon)$ as illustrated in Figure 9. And we consider the length of the time that it takes for the capital stock $K(t)$ to reach $N(\varepsilon)$ from the point K_0 given initially. The paths shown in the phase-diagram, by (40) and (41), have the follow-

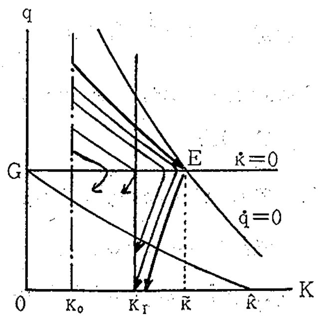

Figure 8

ing properties:

(42) $$\frac{\partial \dot{K}}{\partial q}=\frac{1}{PC''(I-\mu K)}>0$$

(43) $$\frac{\partial \dot{q}}{\partial K}=P[\mu C''(I-\mu K)-F_{KK}(K,L)]>0$$

Then, the following Lemmata hold from (42) and (43).

Lemma 1.

For $t\epsilon[0, T]$, if $\dot{K}>0$, then the smaller $q(t)$ becomes, the longer the time for the capital stock $K(t)$ to reach K_T from K_0.

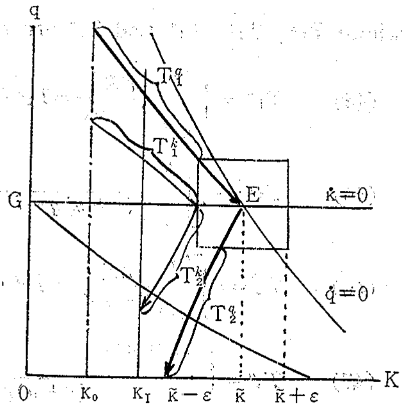

Figure 9

Lemma 2.

For $t\epsilon[0, T]$, if $\dot{q}<0$, then the greater $K(t)$ becomes, the longer the time to reach K_T from K_0.

Thus, from Lemma 1 and Lemma 2, we can understand that the paths lying between T_1^q and T_1^K reach the ε-neighborhood faster than the greater one of the two, T_1^q and T_1^K. Then let T_1 be

$$T_1=\max(T_1^K,\ T_1^q)$$

Next, we shall consider the time to reach K_T or $q(T)=0$ from the ε-neighborhood. Then, the following Lemma holds.

Lemma 3.

If $\dot{K}<0$, then the smaller $q(t)$ becomes, the greater the velocity of $q(t)'s$ decreasing. If $\dot{q}<0$, then the greater $K(t)$ becomes, the smaller the velocity of $q(t)'s$ decreasing.

Thus, in these cases, the paths lying between T_2^K and T_2^q reach K_T or $q(T)=0$ faster than the greater one of the two, T_2^K and T_2^q. Then let T_2 be

$$T_2=\max(T_2^K,\ T_2^q)$$

where T_1^K, T_1^q, T_2^K and T_2^q are shown respectively as follows

(44) $$T_1^K=\int_{\tilde{K}-\varepsilon}^{K_0}\frac{dK}{I-\mu K}=T_1^K(\varepsilon,\ K_0)$$

(45) $$T_1^q=-\int_{\tilde{q}+\varepsilon}^{q_0}\frac{dq}{(r+\mu)q-P\cdot C'(I-\mu K)-PF_K(K,L)}=T_1^q(\varepsilon,q_0)$$

(46) $$T_2^K=\int_{\tilde{K}-\varepsilon}^{K_T}\frac{dK}{I-\mu K}=T_2^K(\varepsilon,K_T)$$

(47) $$T_2^q=\int_{\tilde{q}-\varepsilon}^{max(0,q(T))}\frac{dq}{(r+\mu)q-P\cdot C'(I-\mu K)-PF_K(K,L)}=T_2^q(\varepsilon,q_T)$$

That is, T_1 and T_2 are determined independently of the total period T. Therefore, the optimum paths stay in the ε-neighborhood $N(\varepsilon)$ during the period $T-(T_1+T_2)$. Also, the greater the total period T becomes, the longer the optimum paths stay in $N(\varepsilon)$.

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