STACKELBERG STRATEGY FOR BIMATRIX GAMES

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1. Introduction

Since the original book on games was published in 1944 by J. von Neumann and O. Morgenstern [6], many types of games have been discussed.

In a two-person zero-sum game, the sum of the payoff functions of the two players is equal to zero. That is, the amount that one player gains is equal to the amount that the other loses. Since the objectives of the two players are exactly opposite, there can be neither cooperation nor compromise. In a two-person game with identical goals, that is in a case where the payoff functions for the two players are identical, both players tend to cooperate with each other. The problem to achieve the identical goals is formulated as an optimization problem.

In two-person nonzero-sum games, the objectives of the players are neither exactly opposite nor identical. Several ways of defining a "solution" for the games have been proposed. The "optimal" strategy depends on the rationality assumed by each player. The strategies that have been most investigated are minimax [2], Nash [3], noninferior [4] strategies and so on, each of which has its own characteristics. Besides, a Stackelberg strategy [1] is reasonable when one of the players knows only his own payoff function while the other knows both payoff functions.

In this paper, the Stackelberg strategy is considered and applied to a bimatrix game.

2. Definition of Stackelberg strategy

Definition 2.1 Given a two-person game, where Player 1 wants to maximize a payoff function $J_1(u_1, u_2)$ and Player 2 wants to maximize a payoff function $J_2(u_1, u_2)$ by choosing u_1 and u_2 from admissible strategy sets U_1 and U_2, respectively, the strategy set (u_1^*, u_2^*) is called a *Stackelberg strategy* with Player 2 as leader and Player 1 as follower if for any u_2 belonging to U_2 and u_1 belonging to U_1

$$J_2(u_1^*, u_2^*) \geq J_2(u_1^0(u_2), u_2)$$

where

$$J_1(u_1^0(u_2), u_2) = \max_{u_1} J_1(u_1, u_2)$$

and

$$u_1^* = u_1^0(u_2^*).$$

It is noted that the goal of Player 1 is to maximize J_1 and that of Player 2 is to maximize J_2. A Stackelberg strategy with Player 2 as leader is an optimal strategy for Player 2 if Player 2 announces his move first. If Player 2 chooses any other strategy u_2, then Player 1 will choose a strategy u_1 which maximizes J_1, but the resulting payoff for Player 2 will be less than or equal to the payoff resulting from the Stackelberg strategy with Player 2 as leader. The Stackelberg strategy with Player 2 as leader is attractive for the case where Player 1 does not know the payoff of Player 2, while Player 2 knows both of the payoff functions. By announcing his Stackelberg strategy u_2^* first, Player 2 forces Player 1 to use the Stackelberg strategy u_1^*.

3. Bimatrix game and Nash equilibrium pair

First, let $A=(a_{ij})$ and $B=(b_{ij})$, $1\leq i\leq m$, $1\leq j\leq n$, be the payoff matrices for Player 1 and Player 2, respectively. It is assumed that each row of A is different from others and that each column of B is different from others. If a pure strategy i is chosen by Player 1 and a pure strategy j is chosen by Player 2, then Player 1 receives a_{ij} units and Player 2 receives b_{ij} units. Next, mixed strategies x and y are introduced and the sets of all mixed strategies, X and Y are defined as follow,

$$X=\{x \mid x=(\xi_1,\xi_2,\cdots,\xi_m),\xi_i\geq 0, i=1,\cdots,m, \sum_{i=1}^{m}\xi_i=1\},$$

$$Y=\{y \mid y=(\eta_1,\eta_2,\cdots,\eta_n),\eta_j\geq 0, j=1,\cdots,n, \sum_{j=1}^{n}\eta_j=1\}.$$

If mixed strategies x and y are chosen by Player 1 and Player 2, respectively, then the payoff functions for Player 1 and Player 2 are given by

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\xi_i\eta_j$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}\xi_i\eta_j,$$

respectively.

Incidentally, we consider a Nash equilibrium pair for two-person nonzero-sum games.

Definition 3.1 Let $K_1(x, y)$ and $K_2(x, y)$ be real valued functions on $X\times Y$. A point $(x^0, y^0)\in X\times Y$ is called a Nash equilibrium pair if

$$K_1(x^0,y^0)\geq K_1(x,y^0) \text{ for all } x\in X$$

and

$$K_2(x^0,y^0)\geq K_2(x^0,y) \text{ for all } y\in Y.$$

It is well known that a Nash equilibrium pair exists in mixed strategies for finite nonzero-sum two-person games, i.e., bimatrix games.

Theroem 3.1 [3,5] If Player 1 and Player 2 are non-cooperative with the payoff matrices A and B, respectively, then there exists a mixed strategy $x^0=(\xi_1^0, \xi_2^0, \cdots, \xi_m^0)$ for Player 1 and there exists a mixed strategy $y^0=(\eta_1^0, \eta_2^0, \cdots, \eta_n^0)$ for Player 2 such that

$$K_1(x^0, y^0)=\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\xi_i^0\eta_j^0 \geq \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\xi_i\eta_j^0 = K_1(x, y^0)$$

for all $x \in X$

and

$$K_2(x^0, y^0)=\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}\xi_i^0\eta_j^0 \geq \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}\xi_i^0\eta_j = K_2(x^0, y)$$

for all $y \in Y$.

In general, a Nash equilibrium pair has a meaning for the cases where both players have a common information pattern and choose their strategies at the same time.

4. Stackelberg strategy for bimatrix games

It is noted that the two players are assumed to have the different information pattern from each other and to decide their strategies sequentially. Under this assumption, Stackelberg strategies are considered to be reasonable. Let us consider two kinds of Stackelberg strategies for bimatrix games, say, pure strategies and mixed strategies.

4.1 Pure Stackelberg strategy

First, pure Stackelberg strategies are analyzed. Player 2, the leader, decides a pure strategy j from his strategy set $J=\{1, 2, \cdots, n\}$

and announces it to Player 1. Then Player 1 decides his pure strategy i from his strategy set $I=\{1, 2, \cdots, m\}$ in order to maximize his payoff a_{ij} for fixed j.

Therefore, the follower's pure strategy $i(j)$ should be determined by the relation

$$\max_{i} a_{ij}=a_{i(j)j} \quad \text{for any fixed } j\in J$$

where $i(j)$ is said to be a rational reaction. If Player 2 uses the pure strategy j, then the optimal strategy for Player 1 is the pure strategy $i(j)$. Player 2 must decide j^* to maximize $b_{i(j)j}$. That is, the optimal strategy j^* satisfies the equation

$$\max_{j\in J} b_{i(j)j}=b_{i(j^*)j^*}.$$

When $i(j)$ is not single valued, a set of indices which maximize a_{ij} for fixed j is denoted by $I(j)$. Player 1 can use any element of $I(j)$ to maximize a_{ij}. In this case we must modify $i(j)$ by the relation

$$\min_{i\in I(j)} b_{ij}=b_{i(j)j},$$

because Player 1 may use the strategy i which satisfies the above equation from $I(j)$.

From the above discussion, the pure Stackelberg solution is determined as follows:

Step 1. Find the maximum element $a_{i(j)j}$ from a_{ij}, $i=1, 2, \cdots, m$, for each $j\in J$. If maximum elements are not uniquely determined, then $i(j)$ is the index which minimizes b_{ij} among the index set $I(j)$.

Step 2. Find the index j^* which maximizes $b_{i(j)j}$.

Step 3. Follower's Stackelberg strategy i^* is then determined by the relation

$$i^* = i(j^*).$$

4.2 Mixed Stackelberg strategy

Next, we consider mixed Stackelberg strategies. It is rational for Player 1 to use the mixed strategy $x=(\xi_1, \xi_2, \cdots, \xi_m)$ such that x maximizes $K_1(x, y)$ on X. Then the problem to find an optimal x for given $y \in Y$ is reduced to the following linear programming problem:

$$\text{maximize } K_1(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \xi_i \eta_j$$

subject to

$$\sum_{i=1}^{m} \xi_i = 1, \quad \xi_i \geq 0, \quad i = 1, 2, \cdots, m.$$

In general, an optimal solution of the linear programming problem is given as an extreme point of X or a convex combination of the optimal extreme points. In both cases, there exists at least one index $i(j)$ such that

$$\max_{x \in X} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \xi_i \eta_j = \sum_{j=1}^{n} a_{i(j)j} \eta_j \qquad \text{for any fixed } y \in Y.$$

Let us define m subsets of Y to facilitate the subsequent discussion.

Definition 4.1

$$Y_k = \{y \mid \sum_{j=1}^{n} a_{kj} \eta_j \geq \sum_{j=1}^{n} a_{ij} \eta_j,\ y \in Y, \quad i = 1, 2, \cdots, m\},$$

$$k = 1, 2, \cdots, m.$$

Intuitively, if $y \in Y_k$ and $y \notin Y_i$, $i \neq k$, then it is rational or optimal for Player 1 to use the pure strategy k for the leader's mixed strategy y. The following proposition holds.

Proposition 4.1 Each Y_k is a compact convex polyhedron and

$$Y = \bigcup_{k=1}^{m} Y_k.$$

It is noted that the pure strategy k can be considered to be a particular mixed strategy $e_k=(0,\cdots,0,1,0,\cdots,0)$ with only the k-th element being 1.

Let us pay attention to one of the subsets Y_k in Y. If Player 2 expects that Player 1 always uses the pure strategy k when Player 2 uses the mixed strategy $y\in Y_k$, then Player 2 will take the payoff β_k which is the optimal value of the linear programming problem $LP(k)$:

$$\text{maximize} \sum_{j=1}^{n} b_{kj}\eta_j$$

subject to

$$y\in Y_k=\{y \mid \sum_{j=1}^{n} a_{kj}\eta_j \geq \sum_{j=1}^{n} a_{ij}\eta_j,\ y\in Y,\ i=1,2,\cdots,m\}.$$

Therefore, it is sufficient for Player 2 to choose the minimum β_{k^*} among $\beta_1, \beta_2, \cdots, \beta_n$. However, in actual situations Player 2 cannot always expect that Player 1 uses the pure strategy k for $y\in Y_k$, because an optimal solution y_k of the $LP(k)$ may also belong to Y_i, $i\neq k$. In this case, Player 1 may use the pure strategy i corresponding to Y_i.

To resolve the above difficulty, let us introduce subsets $\mathring{Y}_k$'s of Y.

Definition 4.2

$$\mathring{Y}_k=\{y \mid \sum_{j=1}^{n} a_{kj}\eta_j > \sum_{j=1}^{n} a_{ij}\eta_j,\ y\in Y,\ i\neq k\},\ k=1,2,\cdots,m.$$

For $y\in\mathring{Y}_k$, Player 1 always uses the pure strategy k. It is noted that $\mathring{Y}_k\subseteq Y_k$ and that $\mathring{Y}_k$ is convex but not always compact. Then, $LP(k)$ must be modified as follows,

$$\text{maximize} \sum_{j=1}^{n} b_{kj}\eta_j$$

subject to

$$y \in \mathring{Y}_k.$$

The above optimization problem is not a linear programming problem because the feasible region $\mathring{Y}_k$ is not compact. Therefore, the payoff $\sum_{j=1}^{n} b_{kj}\eta_j$ does not always achieve its maximum value in $\mathring{Y}_k$. Let us define $\mathring{\beta}_k$ analogously to β_k.

Definition 4.3

$$\mathring{\beta}_k = \sup_{y \in \mathring{Y}_k} \sum_{j=1}^{n} b_{kj}\eta_j$$

It is noted that $\mathring{\beta}_k \leq \beta_k$ because $\mathring{Y}_k \subseteq Y_k$. Formally, the problem is reduced to the following problem: find the minimum $\mathring{\beta}_{k^*}$ among $\mathring{\beta}_1$, $\mathring{\beta}_2$, $\cdots$, $\mathring{\beta}_n$.

If the closure of $\mathring{Y}_k$ is equal to Y_k, then $\mathring{\beta}_k = \beta_k$ because $\sum_{j=1}^{n} b_{kj}\eta_j$ is a continuous function of y. On the other hand, if $\mathring{Y}_k$ is a null set but Y_k is not a null set, then the relation $\mathring{\beta}_k = \beta_k$ does not hold. Furthermore, the point y which achieves $\mathring{\beta}_k$ may not exist in $\mathring{Y}_k$. By the definition of supremum there exists y in $\mathring{Y}_k$ such that

$$\sum_{j=1}^{n} b_{kj}\eta_j > \mathring{\beta}_j - \varepsilon \text{ for any } \varepsilon > 0.$$

This means that Player 2 can choose the mixed strategy y in $\mathring{Y}_k$ such that the value $\sum_{j=1}^{n} b_{kj}\eta_j$ is arbitrarily close to $\mathring{\beta}_k$.

It is not easy to obtain Stackelberg strategy for bimatrix games by the procedure mentioned above, because $\mathring{\beta}_k$ is not the optimal value of $LP(k)$. First, note that there exists a bimatrix game which does not have a mixed Stackelberg strategy. This case occurs in the following situation. Let $\max_k \beta_k = \beta_{k^*}$ and $\bar{y} = (\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_n)$ be the optimal solution of $LP(k^*)$. If $\bar{y}$ also belongs to Y_i, $i \neq k^*$, then Player 2 cannot receive the payoff β_{k^*} when Player 1 uses the pure strategy i. Since Player 2 cannot know in advance whether Player 1 uses pure strategy k^* or $i \neq k^*$, this implies that no Stackelberg strategy exists.

To avoid this difficulty, the second approach removes the points which belong to two or more Y_i's. In this case, however, another difficulty arises that the value $\mathring{\beta}_k$ may not be achievable in $\mathring{Y}_k$ since $\mathring{\beta}_k$ is the supremum value, so we may not be able to determine the Stackelberg strategy exactly through this approach. Thus a strategy can only be obtained approximately.

To get an approximate solution, it may be a better way for us to solve the following linear programming problems $LP(\varepsilon;k)$, $k=1,2,\cdots,m$:

$$\text{maximize} \sum_{j=1}^{n} b_{kj}\eta_j$$

subject to

$$y \in Y_k^{\varepsilon}$$

where

$$Y_k^{\varepsilon} = \{y \mid \sum_{j=1}^{n} a_{kj}\eta_j \geq \sum_{j=1}^{n} a_{ij}\eta_j + \varepsilon,\ y \in Y,\ i \neq k\}$$

and $\varepsilon > 0$ is an appropriate constant.

Then the following proposition holds.

Proposition 4.2 Y_k^{ε} is a compact convex polyhedron and $Y_k^{\varepsilon} \subseteq Y_k$.

The optimal value of $LP(\varepsilon;k)$ is also introduced.

Definition 4.4

$$\beta_k^{\varepsilon} = \max_{y \in Y_k^{\varepsilon}} \sum_{j=1}^{n} b_{kj}\eta_j .$$

If it is assumed that Player 1 neglects the difference ε between $\sum_{j=1}^{n} a_{ij}\eta_j$ and $\sum_{j=1}^{n} a_{kj}\eta_j$ when he chooses his strategy, then Y_k^{ε} expresses the region where Player 1 surely uses the pure strategy k. This assumption is accepted in many actual situations. Thus, the approximate Stackelberg strategy is obtained by the following procedure.

Step 1. Solve the $LP(\varepsilon;k), k=1, 2, \cdots, m$, for appropriately chosen $\varepsilon>0$. Let an optimal solution and the optimal value be y_k^ε and β_k^ε, respectively.

Step 2. Find the minimum $\beta_{k^*}^\varepsilon$ among $\beta_1^\varepsilon, \beta_2^\varepsilon, \cdots, \beta_n^\varepsilon$.

Step 3. Let the Stackelberg strategies for Player 2 and Player 1 be $y_{k^*}^\varepsilon$ and k^*, respectively.

Let us call the above Stackelberg strategy "ε-approximate Stackelberg strategy."

The third approach has an advantage over others in the following two points of view.

Proposition 4.3 1. The ε-approximate Stackelberg strategy always exists for sufficiently small $\varepsilon>0$ and can be obtained by solving m $LP(\varepsilon;k)$. 2. If the maximum β_{k^*} of the first approach gives the exact Stackelberg strategy, then it can be obtained by letting $\varepsilon>0$ to zero for β_k^ε.

Proof. 1. It is sufficient to show that $Y_k^\varepsilon \cap Y_i^\varepsilon=\phi$ for $k\neq i$ and that there exist $\varepsilon>0$ and k such that $Y_k^\varepsilon\neq\phi$. Then, at least one β_k^ε is defined and $\min_k \beta_k^\varepsilon$ exists. There exist index k and $y^0=(\eta_1{}^0, \eta_2{}^0, \cdots, \eta_n{}^0)$ in Y such that

$$\sum_{j=1}^{n} a_{kj}\eta_j{}^0 > \sum_{j=1}^{n} a_{ij}\eta_j{}^0 \quad \text{for } i\neq k$$

because $(a_{k1}, a_{k2}, \cdots, a_{kn})\neq(a_{i1}, a_{i2}, \cdots, a_{in})$ by the assumption of the bimatrix game.

If ε is defined by the relation

$$\varepsilon=\frac{1}{2}\left(\sum_{j=1}^{n} a_{kj}\eta_j{}^0-\sum_{j=1}^{n} a_{ij}\eta_j{}^0\right)>0\,,$$

then y^0 belongs to Y_k^ε because

$$\sum_{j=1}^{n} a_{kj}\eta_j{}^0=\sum_{j=1}^{n} a_{ij}\eta_j{}^0+2\varepsilon\geq\sum_{j=1}^{n} a_{ij}\eta_j{}^0+\varepsilon.$$

This proves that $Y_k^\varepsilon\neq\phi$. It is shown that

$$Y_k^\varepsilon\cap Y_i^\varepsilon=\phi \text{ for } k\neq i$$

from the definition of Y_k^ε and Y_i^ε. This completes the proof of the first part of proposition.

2. The exact Stackelberg strategy exists for the two cases that the optimal vector y^* corresponding to β_k* belongs to only Y_k and that y^* is the optimal solution for each $LP(k)$ where k is the index of Y_k to which y^* belongs. In the first case, it holds that

$$\sum_{j=1}^{n} a_{k^*j}\eta_j{}^*>\sum_{j=1}^{n} a_{ij}\eta_j{}^* \text{ for } i\neq k$$

because y^* belongs to only Y_{k^*}. Therefore, y^* also belongs to Y_k^ε where

$$\varepsilon=\frac{1}{2}\left(\sum_{j=1}^{n} a_{k^*j}\eta_j{}^*-\sum_{j=1}^{n} a_{ij}\eta_j{}^*\right).$$

This proves that y^* is also the optimal solution of $LP(\varepsilon;k)$, because $Y_k^\varepsilon\subseteq Y_k$ and y^* is the optimal solution $LP(k)$. In the second case, if δ is a sufficiently small positive number, then any point y in Y such that $|y-y^*|<\delta$ belongs to one of the Y_k's to which y^* belongs, because each function $\sum_{j=1}^{n} a_{kj}\eta_j$ is continuous. Therefore, the above y belongs to one of Y_k^ε's as ε tends to zero. This means that the optimal value β_k^ε of each $LP(\varepsilon;k)$ tends to β_k, respectively, by letting ε to zero, because $\sum_{j=1}^{n} b_{kj}\eta_j$ is also continuous. This completes the second part of proposition.

5. Conclsion

The Stackelberg strategy is applied to finite two-person nonzero-sum games or bimatrix games.

It is shown that there exists a bimatrix game which does not have a mixed Stackelberg strategy.

To determine the Stackelberg strategy in actual situation, ε-approximate Stackelberg strategy is introduced and the procedure for obtaining it is given. Some properties of the ε-approximate Stackelerg strategy are proved.

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