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Group Decision Rules : Characterisations of two basic properties

By Ravindra R. RANADE

Equivalent rationality conditions have been provided for two basic properties used in the proofs of results in social choice theory, including Arrow's impossibility theorem.

1. The two following properties have been extensively used in proving results in the social choice theory. We will name them after their discoverers [see Arrow (1963), Sen (1970)].

Arrow property : (AP) : A set of individuals is almost decisive over an ordered pair of alternatives only if it is decisive over every pair of alternatives.

Sen property : (SP) : A set of individuals is almost semidecisive over an ordered pair of alternatives only if it is semidecisive over every pair of alternatives.

The conditions used in the results are sufficient to guarantee AP or SP but are not necessary for them. It is of some interest to characterise the classes of binary aggregation procedures which satisfy AP and SP respectively. The direct proofs of the results can then be obtained to make the structures transparent. We address ourselves to these characterisations.

2. Let E denote the set of feasible, mutually exclusive alternatives for social choice. It has at least three elements. N, a finite set with cardinali-

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ty $n \geqq 3$, denotes the set of individuals. $\Re$ denotes the set of all logically possible orderings and Ω, the set of all possible reflexive and connected binary relations defined over the set E. $\langle R_i \rangle$ denotes an n-tuple of orderings, one for each individual. R cenotes the social preference relation (SPR). The symmetric and asymmetric components of R_i and R are denoted by I_i and P_i, and I and P respectively.

A *group decision rule* (GDR) is a function $f: \Re^n \rightarrow \Omega$, and will be written as $f(\langle R_i \rangle) = R$.

Two of the most important theorems which are usually proved using AP and SP are the Arrow's impossibility tneorem (AIT) and the axiomatisation of Pareto extension rule (APE). The former states that there is no binary, Paretian and nondictatorial GDR which yields transitive SPR for all elements of $\Re^n$. The latter states that the only GDRs which are binary, Paretian and anonymous such that the SPR is always quasitransitive, are the ones which declare all Pareto incomparable alternatives as socially indifferent. That is to say, $(\exists i, j \in N$ such that $xP_iy \wedge yP_jx) \rightarrow$ xIy. Although in the class of binary GDRs, Pareto criterion (to be taken in its weak form all through these pages) and quasitransitivity are sufficient for AP and SP, they are not necessary. In an unpublished work, Jain (1984) has shown the following condition to be necessary and sufficient for the binary GDRs to satisfy (AP $\wedge$ SP).

Weak Pareto quasitransitivity: (WPQT): $\forall x, y, z \in E$, $\{(xPy \wedge yQz) \veebar (xQy \wedge yPz)\} \rightarrow xPz$, where (and hereafter) aQb denotes $(aP_ib, \forall i \in N)$.

It may be noted that a condition similar to WPQT has been mentioned by Fishburn (1973). The condition WPQT, however, does not serve adequately for revealing the structures of AIT and APE because it does not characterise GDRs satisfying AP and SP separately. Towards these cha-

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racterisations let us define the following.

Restricted Pareto quasitransitivity : (RPQT): $\forall x, y, z \in E$, $\{(xPy \wedge yQz \wedge xI_iy$ for no $i \in N) \veebar (xQy \wedge yPz \wedge yI_iz$ for no $i \in N)\} \rightarrow xPz$.

Limited Pareto quasitransitivity : (LPQT): $\forall x, y, z \in E$, $\{(xPy \wedge yQz \wedge xI_iz$ for no $i \in N) \veebar (xQy \wedge yPz \wedge xI_iz$ for no $i \in N)\} \rightarrow xPz$.

Theorem : A binary GDR, (i) satisfies AP iff it satisfies RPQT and (ii) satisfies SP iff it satisfies LPQT.

Proof : (i) The "if" part follows from the traditional lines of the lemma used in the proof of AIT [see Sen (1970), e. g.]. We will prove the "only if" part. Let AP be satisfied and consider the situation such that $(xPy \wedge yQz \wedge xI_iy$ for no $i \in N)$. Only four orderings can be held by the individuals. 1. xP_iyP_iz, 2. yP_ixP_iz, 3. yP_ixI_iz, 4. yP_izP_ix. If V' denotes the set of individuals who hold 1., then V' is almost decisive over (x, y), given the binary property of the GDR. Thanks to AP, V' is decisive over (x, z). As xP_iz, $\forall i \in V'$, xPz results in the given situation. By a similar argument it can be shown that $(xQy \wedge yPz \wedge yI_iz$ for no $i \in N)$ implies xPz. Therefore AP $\rightarrow$ RPQT.

(ii) Here, again, the 'if' part can be proved along the traditional lines and we will merely indicate the argument here. Let LPQT be satisfied and a set V be almost semidecisive over (x, y). Let z be a third alternative and consider the following preferences. $(\forall i \in V, xP_iyP_iz \wedge \forall j \in N-V, yP_jx \wedge yP_jz)$. As V is almost semidecisive over (x, y), xRy results. Now, if zPx then $(yQz \wedge zPx \wedge yI_ix$ for no $i \in N) \rightarrow yPx$, by LPQT. This is not true and therefore xRz results. As nothing has been said about preferences of $i \in N-V$ over (x, z) and that xP_iz, $\forall i \in V$, the binariness $\rightarrow$ V is semidecisive over (x, z). Using this type of argument, it can be shown that V is semidecisive over every pair of alternatives in E. So, LPQT $\rightarrow$ SP.

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Secondly the 'only if' part. Let SP be satisfied by a binary GDR. Consider a situation with $(xPy \wedge yQz \wedge xI_iz$ for no $i \in N)$.
Only four orderings can be held by the individuals. 1. xP_iyP_iz, 2. xI_iyP_iz, 3. yP_ixP_iz, 4. yP_izP_ix. If V denotes the set of individuals holding 4. then $zRx \rightarrow$ V is almost semidecisive over (z, x). By SP it is semidecisive over (y, x) which implies yRx. But that is not true. Therefore $\sim zRx$ i. e. xPz in the above situation. By a similar argument it can be shown that $(xQy \wedge yPz \wedge xI_iz$ for no $i \in N) \rightarrow xPz$. Hence SP $\rightarrow$ LPQT. Q. E. D.

3. The direct proofs of AIT and APE can be provided in the light of the above result. Blau (1972) discusses AIT and here we discuss APE.

In the class of binary GDRs, the Pareto principle and quasi-transitivity imply that LPQT is satisfied. From the above result we need to confine ourselves only to the GDRs satisfying SP. Consider a binary GDR which does not declare all the Pareto incomparable alternatives as socially indl-fferent and is anonymous. We will show that such a GDR yields non-quasitransitive SPR for some configuration of individual preferences. Suppose not. Consider the following configuration over three alternatives. Individual 1: xP_1yP_1z; 2: yP_2zP_2x and $\forall i \in N$-{1, 2}: zP_ixP_iy. If xRz then 1 is almost semidecisive over (x, z) and given SP and anonymity every individual is semidecisive over every pair of alternatives. This implies that the GDR must declare every pair of Pareto incomparable alternatives as socially indifferent. Therefore $\sim xRz$, i. e. zPx must result. By a similar argument involving individual 2, xPy can be concluded. Quasitransitivity $\rightarrow zPy$ results. This implies that the set {1, 2} is not almost semidecisive over $\{y, z\}$ and thanks to SP, not almost semidecisive over any ordered pair of alternatives. Now, consider another situation: $\forall i \in$ {1, 2}, xP_iyP_iz, 3: yP_3zP_3x and $\forall\ j \in N$-{1, 2, 3}, zP_jxP_jy. In this situation,

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by a reasoning similar to above, we conclude that the set {1, 2, 3} is not almost semidecisive over any ordered pair of alternatives. By considering the individuals one by one from the set N, given the finiteness of the set N, we conclude that the Pareto principle is violated. This contradiction esteblishes that our supposition about quasitransitivity must be wrong.

The direct proof of the APE heightens the use of SP and the finiteness of the set N in the result. It alno vividly brings out how all the reasonable GDRs (Paretian and democratic) like the majority rules face the problem of irrational SPR for the individual preferences of the type used in the well known paradox of voting.

4. Direct proofs of AIT and APE are not the only achievements possible of the above characterisations of AP and SP. All the results in the social choice theory normally make use of a cartain degree of neutrality and monotonicity in the decisive powers of groups of individuals. The close connection between the rationality properties of social preferences and the neutrality and the monotonicity of the GDRs has always been the undercurrent in theorisation, although not always explicitly atated as such. The characterisations of this paper will be helpful in making that undercurrent visible and making the results in social choice theory structurally transparent.

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