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Some Aspects of Statistical Inference of Weibull Parameters with Wide Applicability in Reliability-Based Design

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1 Introduction

Practical machines and structures are usually subjected to randomly varying external loads, and the strength of identical components will never be the same, even under the same loading conditions. In other words, both the load and the strength are of an indeterministic nature [1]～[5]. In addition, a variety of uncertainty factors will inevitably arise in the processes of their construction and maintenance. Engineering uncertainties have in general, as is well known, the following wide range of meanings [6] :

(1) randomness - uncertainty due to inherently random nature.

(2) fuzziness - uncertainty caused by that the object is too complicated to understand, or by insufficient knowledge.

(3) ambiguity - uncertainty contained in natural language.

(4) vagueness - uncertainty included in, for instance, image processing.

(5) imprecision - uncertainty due to lack of information.

(6) generality - uncertainty due to multi-meanings or multi-interpretations for the object.

Among these uncertainties, the most essential and important is unquestionably the randomness which is the very objective the theory of probability and statistics deals with, and the present study also focusses upon.

In order to perform rational design and maintenance, these uncertainties have to be properly evaluated on a probabilistic basis. This is why reliability should be emphasized in the rational design [7]～[12]. The late Professor A. M. Freudenthal first introduced his well-known concept of failure probability to handle this problem in 1946. Following his creative research work, a number of studies have been carried out in the field where safety and reliability both play an important role. Needless to say, safety and reliability play a crucial role in a variety of engineering fields such as material science, mechanical engineering, civil and architectural engineering, naval architecture, aeronautical and space engineering and nuclear engineering, to name but a few. The notion of structural safety and reliability has become of crucial importance, which is reflected by increasing societal concern to a considerable extent. Recently, a number of research works in the field of structural safety and reliability have been published [13]～[95].

In 1969, the first International Conference on Structural Safety and Reliability (abbreviated by ICOSSAR'69), was formed and held in the USA under the chairmanship of the late Professor A. M. Freudenthal of George Washington University (formerly he was at Columbia University), in cooperation with Professor M. Shinozuka of Columbia University (at present he is at Princeton University), Professor A. H-S. Ang of University of Illinois (he is now at University of California, Irvine), and the late Professor Emeritus I. Konishi at Kyoto University, Japan. The ICOSSAR conference has grown up to draw much attention from those researchers and practicing professionals studying and working in the field of structural reliability and probabilistic mechanics. The successive second international conference (ICOSSAR '77) was held in Germany in 1977, and after that time, in the light of promting the societal concern, the conference has been decided to be held every four years in different part of the world. The third conference (ICOSSAR '81) was held in Norway in 1981. In the fourth conference (ICOSSAR '85) held in Japan in 1985, where one of the authors served as Chairman of Conference Organizing Committee, there were nearly 500 participants with presentation of more than 200 papers in total. Furthermore, the first Japan Conference on Structural Safety and Reliability (JCOSSAR '87) was held in December 1987 under the auspices of the Japan Science Council.

Last summer the fifth conference (ICOSSAR '89) was held in San Francisco, California on 7-11 August 1989, where more than 500 persons participated from more than 20 countries and nearly 400 papers on structural safety and reliability were presented with much eager discussions. This is really one of the evidences that the importance and significance of structural safety and reliability come to be fully recognized all over the world. At the closing session of ICOSSAR '89, the announcement was made that the next conference (ICOSSAR '93) would be held in Innsbruck, Austria in 1993.

As stated earlier, most of machines and structures will fail due to the repetition of varying loads, which is called fatigue. Hence, in the practical design, the correct prediction of the fatigue strength or fatigue life of structural components is indispensable under actual service conditions [1], [2]. However, the fatigue strength or fatigue life of identical components will never be the same even under the same loading conditions. That is, it has an inherent scatter. Hence, it becomes of crucial importance to clarify the type of distribution it will follow. Assumed that such failure physics [96] as the mechanism of fatigue failure is made clear, the distribution of the fatigue strength or fatigue life could be theoretically derived. At present, however, we cannot but take the method to predict, at first, empirically the failure probability model to be fitted reasonably well to the obtained data and then to estimate the statistical parameters of the model to be used in the reliability-based design or analysis.

As is well known in this respect, the fatigue life is often successfully fitted to a Weibull distribution [97], which is characterized by two (the shape and scale parameters) or three parameters (the shape, scale and location parameters). The location parameter is often assumed zero in a sense that failure might occur on the moment of the beginning of service. This is the case of a two-parameter Weibull distribution on which the present study mainly focusses [98], [99].

Assuming that the fatigue strength or life follows a two-parameter Weibull distribution, the most important work is how to estimate their distribution parameters. In this connection, the present study concerns with so-called statistical inference in detail, that is, how to estimate statistical parameters (the shape and scale parameters) of the distribution from available data. The reliability analysis, for instance, to determine the design safe life based upon the given reliability level is performed with the aid of estimated values of parameters. Hence, the statistical inference procedure discussed in the present paper becomes of crucial importance in the reliabslity-based design of machines and structures.

2 Order Statistics and Notion of TTFF

In general, more information will be extracted, from a set of data randomly sampled, by sorting them systematically, for instance, in order of magnitude. In the analysis of the distribution of life or time to failure, TTFF (time to first failure) or TTLF (time to last failure) is more reasonable than the central tendancy of randomly extracted data. In this respect, this section provides in detail the basic notion of the order statistics.

Let $T^{(1)}$, $T^{(2)}$, $\cdots$, and $T^{(n)}$ be the random sample of size n taken from the population of the failure life $T(T \geq 0)$ having a continuous probability density function $f(t)$. By arranging these sample random variables in ascending order of magnitude, we get

$$T_1 \leq T_2 \leq \cdots \leq T_n$$

where $T_j(j=1, 2, \ldots, n)$ is called the j-th order statistic of size n. As is easily known, T_j thus defined is also considered to be a random variable.

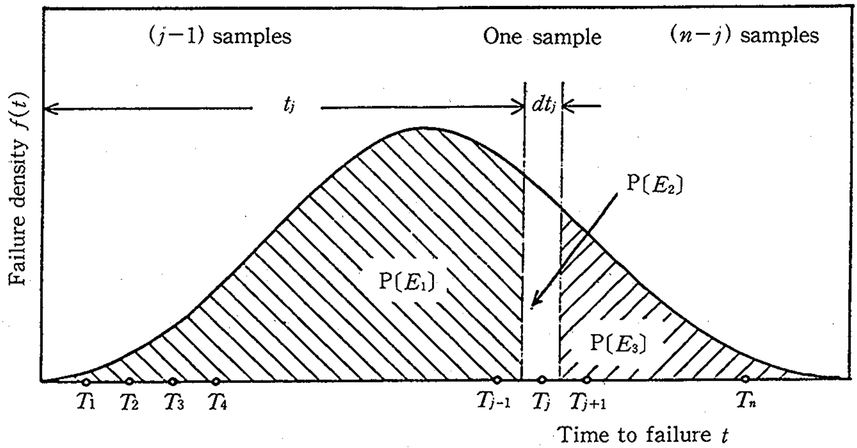


Fig. 2. 1 Explanatory figure to find out the probability of the occurrence of the j-th order statistic.

At this point, let us think of the probability that T_j takes on a value between t_j and t_j+dt_j. As shown in Fig. 2. 1, this is given as the probability of the joint event of E_1, E_2 and E_3, as follows:

$$f_{j;n}(t_j)dt_j = \mathrm{P}[t_j \leq T_j \leq t_j+dt_j]$$
$$= \frac{n!}{(j-1)!1!(n-j)!}\{\mathrm{P}[E_1]\}^{j-1}\{\mathrm{P}[E_2]\}^1\{\mathrm{P}[E_3]\}^{n-j} \qquad (2.1)$$

where E_1 is the event that $(j-1)$ sample elements, T_1, T_2, ..., T_{j-1}, lie in the time interval $(0, t_j)$, E_2 the event that T_j exists in the interval (t_j, t_j+dt_j), and E_3 the event that $(n-j)$ sample elements, T_{j+1}, T_{j+2}, ..., T_n, lie in the interval (t_j+dt_j, ∞).

Since the probability of the occurrence of each event E_1, E_2 or E_3 is given as

$$\mathrm{P}[E_1] = \int_0^{t_j} f(\xi)d\xi = F(t_j)$$
$$\mathrm{P}[E_2] = \int_{t_j}^{t_j+dt_j} f(\xi)d\xi = F(t_j+dt_j)-F(t_j) \cong f(t_j)dt_j$$
$$\mathrm{P}[E_3] = \int_{t_j+dt_j}^{\infty} f(\xi)d\xi = 1-F(t_j+dt_j)$$
$$\cong 1-F(t_j)-f(t_j)dt_j \qquad (2.2)$$

we get the following relationship by substituting Eq. (2. 2) into Eq. (2. 1):

$$f_{j;n}(t_j)dt_j = \frac{n!}{(j-1)!(n-j)!}\{F(t_j)\}^{j-1}\{f(t_j)dt_j\}\{1-F(t_j)-f(t_j)dt_j\}^{n-j}$$
$$= \frac{n!}{(j-1)!(n-j)!}\{F(t_j)\}^{j-1}\{1-F(t_j)\}^{n-j}f(t_j)dt_j$$
$$+(\text{higher order terms of } dt_j \text{ than the second order}) \qquad (2.3)$$

Taking the limitation as $dt_j \to 0$ after dividing the both sides of Eq. (2. 3) by dt_j, we get

$$f_{j;n}(t_j) = \frac{n!}{(j-1)!(n-j)!}\{F(t_j)\}^{j-1}\{1-F(t_j)\}^{n-j}f(t_j) \qquad (2.4)$$

which is nothing but the density function of T_j. The cumulative distribution function $F_{j;n}(t_j)$ of T_j can be computed as

$$F_{j;n}(t_j) = \int_0^{t_j} f_{j;n}(\xi)d\xi \tag{2.5}$$

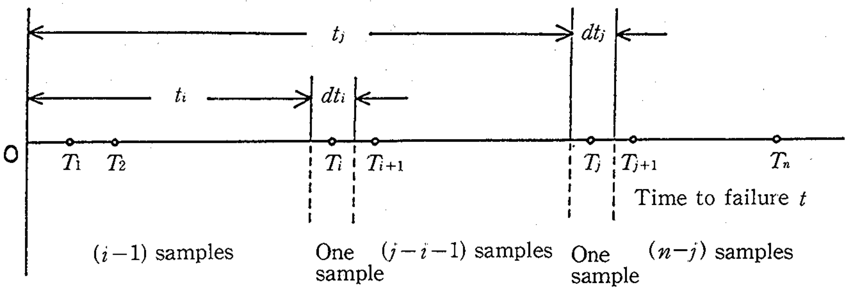


Fig. 2. 2 Schematic explanation to find out the joint density function of the i-th and the j-th order statistics.

In the next step, let us consider the joint probability density function $f(t_i, t_j)$(where $0 \leq t_i \leq t_j$) of the i-th and the j-th order statistics, T_i and T_j (where $1 \leq i \leq j \leq n$). In reference to Fig. 2. 2, this joint density function can be obtained, by use of the notion of a polynominal distribution, as

$$\begin{aligned} f(t_i, t_j)dt_i dt_j &= \mathrm{P}[t_i \leq T_i \leq t_i + dt_i,\ t_j \leq T_j \leq t_j + dt_j] \\ &= \frac{n!}{(i-1)!1!(j-i-1)!1!(n-j)!} \\ &\quad \times P_1^{i-1} P_2^1 P_3^{j-i-1} P_4^1 P_5^{n-j} \end{aligned} \tag{2.6}$$

where

P_1 = the probability that $(i-1)$ elements lie in $(0, t_i)$

$$= \int_0^{t_i} f(\xi)d\xi = F(t_i)$$

P_2 = the probability that T_i lies in the interval $(t_i, t_i + dt_i)$

$$= F(t_i + dt_i) - F(t_i) \cong f(t_i)dt_i$$

P_3 = the probability that $(j-i-1)$ elements lie in $(t_i + dt_i, t_j)$

$$= F(t_j) - F(t_i + dt_i) \cong F(t_j) - F(t_i) - f(t_i)dt_i$$

P_4 = the probability that T_j lies in $(t_j, t_j + dt_j)$

$$= F(t_j + dt_j) - F(t_j) \cong f(t_j)dt_j$$

P_5 = the probability that $(n-j)$ elements lie in $(t_j + dt_j, \infty)$

$$= 1 - F(t_j + dt_j) \cong 1 - F(t_j) - f(t_j)dt_j \tag{2.7}$$

Substituting Eq. (2.7) into Eq. (2.6), deviding the both sides by $dt_i dt_j$, and finally taking the limitation as $dt_i \to 0$ and $dt_j \to 0$, we get the following joint density:

$$f(t_i, t_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times \{F(t_i)\}^{i-1}\{F(t_j) - F(t_i)\}^{j-i-1} \times \{1 - F(t_j)\}^{n-j} f(t_i) f(t_j) \tag{2.8}$$

where $0 \leq t_i \leq t_j < \infty$.

In what follows, some of important notions associated with order statistics are discussed briefly which are of considerable importance in the field of reliability engineering.

2.1 Distributions of TTFF, TTSF and TTLF

The probability density function $f_{1;n}(t_1)$ and the cumulative distribution function $F_{1;n}(t_1)$ of the time to first failure (the minimum life) T_1, that is, TTFF, can be obtained by replacing j by $1(j=1)$ in Eqs. (2.4) and (2.5) as

$$\left.\begin{aligned} f_{1;n}(t_1) &= n\{1 - F(t_1)\}^{n-1} f(t_1) \\ F_{1;n}(t_1) &= 1 - \{1 - F(t_1)\}^n \end{aligned}\right\} \tag{2.9}$$

Similarly, the density and the cumulative distribution functions of both the second minimum life TTSF (the time to second failure) and the maximum life TTLF (the time to last failure) can be given, by putting $j = 2$ and $j = n$, respectively, as

$$\left.\begin{aligned} f_{2;n}(t_2) &= n(n-1)F(t_2)\{1 - F(t_2)\}^{n-2} f(t_2) \\ F_{2;n}(t_2) &= 1 - \{1 - F(t_2)\}^n - nF(t_2)\{1 - F(t_2)\}^{n-1} \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} f_{n;n}(t_n) &= n\{F(t_n)\}^{n-1} f(t_n) \\ F_{n;n}(t_n) &= \{F(t_n)\}^n \end{aligned}\right\} \tag{2.11}$$

As stated earlier, since the following relationship holds between the cumulative distribution function $F(t)$ and the reliability function $R(t)$:

$$R(t)=1-F(t) \tag{2.12}$$

the reliability function for each case mentioned above takes the following form:

Reliability function of TTFF:

$$R_{1;n}(t_1)=\{R(t_1)\}^n \tag{2.13}$$

Reliability function of TTSF:

$$R_{2;n}(t_2)=n\{R(t_2)\}^{n-1}\left\{1-\left(\frac{n-1}{n}\right)R(t_2)\right\} \tag{2.14}$$

Reliability function of TTLF:

$$R_{n;n}(t_n)=1-\{1-R(t_n)\}^n \tag{2.15}$$

It should be mentioned at this point that the abovementioned quantities, say TTFF, need to be treated as random variables. Therefore, the observation both of the mean as the central tendancy of the variate, and of the variance as a measure of scatter becomes of much interest. For example, assuming that the distribution $F(t)$ of the population follows a two-parameter Weibull distribution [100] with the shape parameter α and scale parameter β which will be discussed in detail in the following sections, the means and variances of TTFF and TTSF among n elements can be calculated as follows:

MTTFF (mean time to first failure):

$$\mathrm{E}[T_1]=\beta\left(\frac{1}{n}\right)^{1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right)=\left(\frac{1}{n}\right)^{1/\alpha}\times(\text{MTTF or MTBF}) \tag{2.16}$$

Variance of TTFF:

$$\mathrm{Var}[T_1]=\left(\frac{1}{n}\right)^{2/\alpha}\beta^2\left[\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma^2\left(1+\frac{1}{\alpha}\right)\right] \tag{2.17}$$

MTTSF (mean time to second failure):

$$\mathrm{E}[T_2]=\left\{n\left(\frac{1}{n-1}\right)^{1/\alpha}-(n-1)\left(\frac{1}{n}\right)^{1/\alpha}\right\}\beta\Gamma\left(1+\frac{1}{\alpha}\right)$$

$$= \left\{ n\left(\frac{1}{n-1}\right)^{1/\alpha} - (n-1)\left(\frac{1}{n}\right)^{1/\alpha} \right\} \times (\text{MTTF or MTBF}) \qquad (2.18)$$

Variance of TTSF:

$$\mathrm{Var}[T_2] = \beta^2\left[\left\{ n\left(\frac{1}{n-1}\right)^{2/\alpha} - (n-1)\left(\frac{1}{n}\right)^{2/\alpha} \right\}\Gamma\left(1+\frac{2}{\alpha}\right) - \left\{ n\left(\frac{1}{n-1}\right)^{1/\alpha} - (n-1)\left(\frac{1}{n}\right)^{1/\alpha} \right\}\Gamma^2\left(1+\frac{2}{\alpha}\right)\right] \qquad (2.19)$$

In this way, such statistical quantities as expected value and variance of, say TTFF, can be computed based upon both the distribution parameter of the population and the sample size. MTTFF, which is the central tendancy of TTFF, is obtained, as shown in Eq. (2.16), by multiplying the central tendancy of the population (that is, MTTF or MTBF) by the factor $(1/n)^{1/\alpha}$, where n represents the sample size and α the shape parameter. However, since the true value of each distribution parametr of the population is usually unknown, its estimate from a sample of size n has to be utilized, which might cause an estimation error in the practical application.

2.2 Distribution of Range

The range W is defined as the difference between the maximum value T_n and the minimum T_1 among a random sample of size n. That is,

$$W = T_n - T_1 \qquad (2.20)$$

Knowing the distribution of the range W is equal to get the distribution of the maximum width of scatters of all the samples drawn, and, consequently, is of much significance. The distribution of W can be easily obtained with the aid of the joint probability density of T_1 and T_n. By putting i = 1 and $j = n$ in Eq. (2.8), the joint probability density is given as

$$f(t_1, t_n) = \frac{n!}{0!(n-2)!0!}\{F(t_1)\}^0\{F((t_n)-F(t_1)\}^{n-2}\{1-F(t_n)\}^0 f(t_1)f(t_n)$$
$$= n(n-1)\{F(t_n)-F(t_1)\}^{n-2}f(t_1)f(t_n) \qquad (2.21)$$

By applying variable transformation from (T_1, T_n) to (T_1, W) such that

$T_1 = T_1$; $W = T_n - T_1$

with the Jacobian [101] of the transform in the following form:

$$\frac{\partial(t_1,\ t_n)}{\partial(t_1,\ w)} = \left\{\frac{\partial(t_1,\ w)}{\partial(t_1,\ t_n)}\right\}^{-1} = \begin{vmatrix} \dfrac{\partial t_1}{\partial t_1} & \dfrac{\partial w}{\partial t_1} \\ \dfrac{\partial t_1}{\partial t_n} & \dfrac{\partial w}{\partial t_n} \end{vmatrix}^{-1} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}^{-1} = 1 \qquad (2.22)$$

the joint probability density $f(t_1,\ w)$ of T_1 and W can be given as follows:

$$f(t_1,\ w) = f(t_1,\ t_n)\left|\frac{\partial(t_1,\ t_n)}{\partial(t_1,\ w)}\right|$$

$$= n(n-1)\{F(t_1+w)-F(t_1)\}^{n-2}f(t_1)f(t_1+w) \qquad (2.23)$$

Therefore, the probability density function of the range W, $f_W(w)$, is obtained as the marginal distribution [102] by integrating Eq. (2.23) over the whole domain with respect to t_1. It should be noted that t_1 takes on a positive value since T_1 is the time to failure.

$$f_W(w) = \int_0^\infty f(t_1,\ w)dt_1$$

$$= n(n-1)\int_0^\infty \{F(t_1+w)-F(t_1)\}^{n-2}f(t_1)f(t_1+w)dt_1 \qquad (2.24)$$

Further application can be exemplified easily. For example, the range execpt both extremal values T_1 and T_n in a sample of size n may be determined in a similar way. However, the detailed discussion is omitted here for lack of space.

2.3 Distribution of Frequency

Let $F(T_j)$ be the probability that the random variable T of the population becomes smaller than the j-th order statistic T_j such that

$$F(T_j) = \mathrm{P}[T \leq T_j] \equiv F_j$$

The quantity F_j is also considered as a random variable, and is called the distribution of the cumulative frequency. The probability density function of F_j can be derived in the following form. First, apply the variable transformation $T_j \rightarrow F_j$ such that

$$F_j = F(t_j) = \int_0^{t_j} f(\xi)d\xi \tag{2.25}$$

Then the density function of F_j can be given as

$$f_{F_j}(F_j) = f_{j;n}(t_j)\left|\frac{dF_j}{dt_j}\right|^{-1}$$

$$= \frac{n!}{(j-1)!(n-j)!}\{F(t_j)\}^{j-1}\{1-F(t_j)\}^{n-j}f(t_j)\cdot\frac{1}{f(t_j)}$$

$$= \frac{n!}{(j-1)!(n-j)!}F_j^{j-1}(1-F_j)^{n-j} \tag{2.26}$$

where $0 \leq F_j \leq 1$. The expected value of F_j, denoted by $E[F_j]$, can be computed with the aid of Eq. (2.26) as

$$E[F_j] = \int_0^1 F_j f_{F_j}(F_j)dF_j$$

$$= \frac{n!}{(j-1)!(n-j)!}\int_0^1 F_j^j(1-F_j)^{n-j}dF_j$$

$$= \frac{n!}{(j-1)!(n-j)!}B(j+1,\ n-j+1)$$

$$= \frac{n!}{(j-1)!(n-j)!}\cdot\frac{\Gamma(j+1)\Gamma(n-j+1)}{\Gamma[n+2)}$$

$$= \frac{j}{n+1} \tag{2.27}$$

where $\Gamma(\cdot)$ is a Gamma function, and $B(\cdot,\ \cdot)$ is a Beta function.

As shown in Eq. (2.27), the probability that the random variable T of the population is smaller than the j-th order statistic T_j of size n, namely, the expected value of the distribution of the cumulative frequency at T_j becomes $j/(n+1)$ in place of j/n. This is the reason to take the plotting position of the j-th order statistic t_j as $F(t_j) = j/(n+1)$ in the mean rank method [103]. In general, the expected value of $\{F_j\}^r$ is given as

$$E[\{F_j\}^r] = \frac{n!(j+r-1)!}{(n+r)!(j-1)!} \tag{2.28}$$

3 Major Failure Models and Associated Distributions

Generally speaking, strength of the material can be considered as the resistance against external stresses. Let S be the external stress and R the internal resistance of the material against it. Then, the failure may be defined as

$$\{R \leq S\} \tag{3.1}$$

This definition of failure is called the stress-strength model. In the case that R or S, or both are considered as random variables, the event that Eq. (3.1) holds may become random and the probability of failure, p_f, is given as follows:

$$p_f = \mathrm{P}[R \leq S] \tag{3.2}$$

Both R and S are generally random variables, and the statistical properties of S can be obtained based upon observations. On the other hand, those of R can be obtained through replication tests. In both cases, obtained data need to be usually processed on a statistical basis, and hence mathematical statistics plays an important role in this respect. In determining the distribution of S or R, there might be some cases where the probability theory itself plays a crucial role as can be seen in applying a normal distribution with the aid of the central limit theorem. Also there might be some other cases to introduce a suitable type of probability model to explain failure phenomena of concern through the empirical observations. In the latter cases, statistical inference plays an indispensable role since the validity of the model needs to be evaluated based upon the comparison between the model distribution and the empirical data obtained by observation or experiment. Parameters of the distribution, either derived from the theory of probability itself or obtained from the assumed model, must be estimated with the aid of statistical treatment of the observed data. In what follows, some typical failure models and associated distributions are briefly discussed on the basis of the statistical approach.

3.1 Pattern of Failure Rate Function

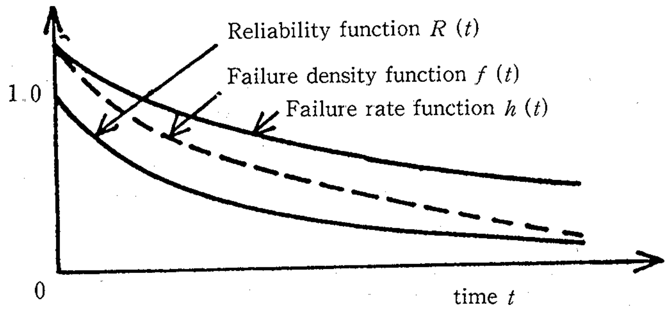
In the reliability analysis of an item, the failure rate function $h(t)$ plays a very important role, for this is directly connected to the probability model of failure. The shape of $h(t)$ as a function of time can be categorized into three basic kinds ; DFR(decreasing failure rate), CFR(constant failure rate) and IFR(increasing failure rate) types described as follows :

(a) DFR type The function $h(t)$ assumes a decreasing value with a lapse of time. This means that, in early time of service, defective parts will fail because of high rate of failure. Therefore, preventive maintenance is of no use since failure rate decreases with increasing time. Of importance is the procedure to remove, before the service, parts of high failure rate with the aid of those techniques such as screening, aging for stabilization and debugging operations and consequently to use remaining parts of good quality. The temporal variability both of the failure density function $f(t)$ and of the reliability function $R(t)$ is schematically represented in Fig. 3.1.

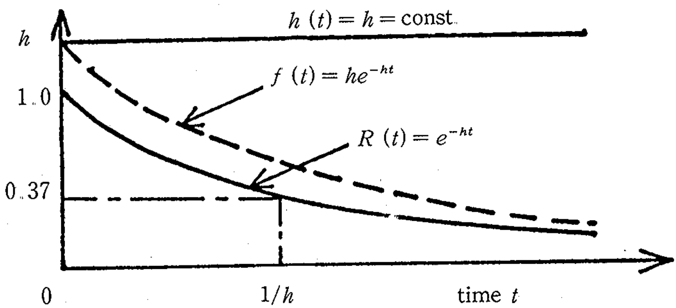
(b) CFR type This is typical in chance failure period for items composed of many parts, where $h(t)$ takes on a constant value and failure is caused completely by chance.

(c) IFR type Failures will occur intensively after a certain amount of service time due to degradation caused by wear and/or fatigue. Preventive maintenance immediately before failure is undoubtedly effective to protect items from failure in advance.

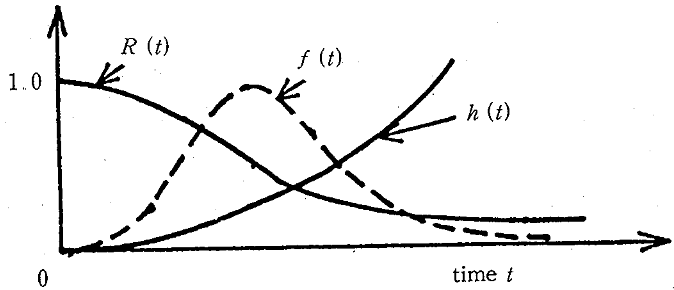
In general, the failure rate function of an item in the non-repair system composed of a large number of elements is represented, as shown in Fig. 3.2, by the shape similar to the cross-section of a western bathtub. That is,



(a) DFR pattern



(c) IFR pattern



(b) CFR pattern

Fig. 3. 1 Patterns of the failure rate function $h(t)$.

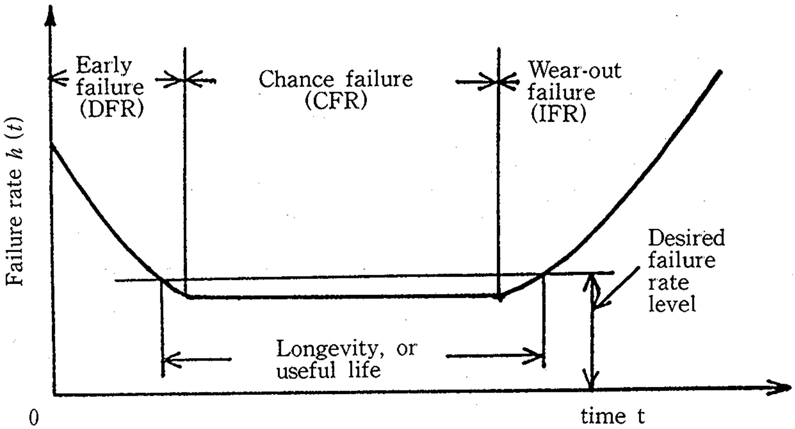


Fig. 3. 2 Typical pattern of failure rate (bathtub curve) for an item without maintenance.

in the early stage of service, there exists early failure period with DFR type of failure rate function, which is caused by defects in the production process and misuse for service environment. Early failure period is followed by chance failure period with CFR type of constat $h(t)$, which comes from the accumulation of various causes of failures of structural components. The final stage is wear-out failure period where failure rate rapidly increases due to cumulative damage by wear and/or fatigue. It is a standard practice to choose the value of failure rate in chance failure period lower than the prescribed. The longevity or useful life is the length of period with actual failure rate of an item being kept lower than the prescribed. For the item in the repair system, the useful life can be extended by applying preventive and corrective maintenance so as to reduce the value of $h(t)$ less than the given.

3. 2 Chance Failure Model and Exponential Distribution

The first interesting failure model is the chance failure model. Suppose that R assumes a constant value. Let $F_S(s)$ be the distribution function of the random variable S, and the probability that failure does not occur within unit time, that is, $P[S < R]$, can be expressed as

$$P[S < R] = P[S \leq R] = F_S(R) \tag{3.3}$$

Conversely, the probability of failure can be given as

$$P[S \geq R] = 1 - F_S(R) = h \tag{3.4}$$

At this point, assume that the above failure probability is kept constant during the entire service time n. Then the probability that failure never occurs throughout the service time is represented as

$$R_e(n) = (P[S < R])^n = \{F_S(R)\}^n = (1-h)^n \tag{3.5}$$

When n becomes large enough, the above equation reduces to

$$R_e(n) = (1-h)^n = (1-nh/n)^n \cong \exp(-nh) \tag{3.6}$$

n may be replaced by time t, and therefore,

$$R_e(t) \cong \exp(-ht) \tag{3.7}$$

h is usually called the failure rate which represents the probability that fracture or failure occurs within unit time. This model is also applicable to the case of random variable R when the failure rate h in unit time never changes at all over the entire service time. $R_e(t)$ is called the reliability function for service time t in chance failure or an exponential distribution, which is the probability that fracture or failure never occurs during this period.

3.3 Proportional Effect Model and Log-Normal Distribution

Of next interest is the proportional effect model from which the log-normal distribution can be derived. Now consider a physical process wherein failure is due to fatigue cracks [104].

Let $X_1 < X_2 < \cdots < X_n$ be a sequence of random variables that denote the size of a fatigue crack at successive stages of its growth. A proportional effect model can be assumed for the growth of these cracks. This implies that the crack growth increment at stage i, $X_i - X_{i-1}$, is randomly proportional to the size of the crack at stage $i-1$, X_{i-1}, and that the material fails when the crack size reaches X_n.

Let $X_i - X_{i-1} = \xi_i X_{i-1}$, $i = 1, 2, \ldots, n$, where ξ_i, the constant of proportionality, is a random variable. The initial size of the crack, X_0, can be interpreted as the size of minute flaws, voids and the like in the material. ξ_i's are assumed to be independently distributed random variables that need not have a common distribution for all i's. Thus,

$$\sum_{i=1}^{n} \xi_i = \sum_{i=1}^{n} \frac{X_i - X_{i-1}}{X_{i-1}} = \sum_{i=1}^{n} \frac{\Delta X_{i-1}}{X_{i-1}} \tag{3.8}$$

If the increment, $X_i - X_{i-1} = \Delta X_{i-1}$, is small at each step, and in the limit, as $\Delta X_{i-1} \to 0$, and n becomes large, it follows that

$$\sum_{i=1}^{n} \xi_i = \int_{X_0}^{X_n} \frac{1}{X} dX = \ln X_n - \ln X_0, \text{ that is}$$

$$\ln X_n = \sum_{i=1}^{n} \xi_i + \ln X_0 \tag{3.9}$$

Since ξ_i's, by assumption, are independently distributed random variables, by the central limit theorem, it follows that they converge in distribution to a normal distribution. Thus $\ln X_n$, the life length of the material, for large n, is asymptotically normally distributed with mean μ and standard deviation σ, and hence X_n has a log-normal distribution.

The statistical properties of a log-normal distribution are given as follows:

Mean:

$$\mu_X = \exp(\mu + \sigma^2/2) \tag{3.10}$$

Variance:

$$\sigma_X^2 = \exp(2\mu + \sigma^2)\{\exp(\sigma^2) - 1\} \tag{3.11}$$

3.4 Weakest Link Model and Weibull Distribution

The third interesting model is the weakest link model. Even in a simple tensile test of round-bar specimen, tensile strength varys from sample to sample. The weakest part of a round-bar specimen is considered to fail since strength may have spatial variation. Hence, the strength distribution may be understood as that of the minimum value.

At this point, assume that the material is composed of n independent elements, and let $F(x)$ be the identical distribution of strength X of each element. In this case, the minimum value distribution $G_n(x)$ among n elements, each of which has the same distribution function $F(x)$, can be represented as follows:

$$G_n(x) = 1-\{1-F(x)\}^n \tag{3.12}$$

Supposing that the minimum value of strength, γ, exists, then $F(x)$ can be defined over the domain $x \geq \gamma$, with $F(\gamma) = 0$. Further, with the assumption such that

$$f(\gamma) = F'(\gamma) = 0,\ f^{(i)}(\gamma) = 0,\ \{i = 1, 2, \ldots, \alpha-2\} \tag{3.13}$$

where α is a positive constant, and by utilizing Taylor series expansion of $F(x)$ around $x = \gamma$ such that

$$F(x) = \frac{(x-\gamma)^\alpha}{\alpha!}\cdot f^{(\alpha-1)}(\gamma)+\frac{(x-\gamma)^{\alpha+1}}{(\alpha+1)!}\cdot f^{(\alpha)}\{\gamma+\theta(x-\gamma)\} \tag{3.14}$$

the following approximation can be made for $|f^{(\alpha)}\{\gamma+\theta(x-\gamma)\}| < M$:

$$\begin{aligned} |n\cdot\ln\{1-F(x)\}+z^\alpha| &= \left|-n\cdot F(x)+n\cdot\frac{(x-\gamma)^\alpha}{\alpha!}\cdot f^{(\alpha-1)}(\gamma)\right| \\ &= \left|n\cdot\frac{(x-\gamma)^{\alpha+1}}{(\alpha+1)!}\cdot f^{(\alpha)}\{\gamma+\theta(x-\gamma)\}\right| \\ &\leq \frac{z^{(\alpha+1)}}{n^{1/\alpha}}\cdot\frac{M}{(\alpha+1)!}\cdot\left[\frac{\alpha!}{f^{(\alpha-1)}(\gamma)}\right] \end{aligned} \tag{3.15}$$

where

$$\beta = \left[\frac{nf^{(\alpha-1)}(\gamma)}{\alpha!}\right]^{-1/\alpha} \tag{3.16}$$

$$x-\gamma = \beta z \tag{3.17}$$

In the above, the following approximation is also introduced in the vicinity of $x=\gamma$:

$$\ln\{1-F(x)\} \cong -F(x)$$

Since the right-hand side of the inequality (3.15) approaches zero when n becomes large, it follows that $\ln\{1-F(x)\}^n \cong -z^\alpha$. Consequently,

$$G_n(x) \to G(x) = 1-\exp(-z^\alpha) = 1-\exp\left[-\left(\frac{x-\gamma}{\beta}\right)^\alpha\right] \qquad (3.18)$$

The above distribution is called a three-parameter Weibull distribution [100] which is frequently used in the analysis of strength or life distribution of the material. The parameters α, β and γ are called the shape, scale and location parameter, respectively.

For the sake of ease of treatment as well as of the fact that fracture might occur on the moment of the beginning of service, the location parameter γ may be regarded as zero in what follows [105]. However, it should be noted that this treatment may not cause any lack of generality of discussion. In this case, the model is called a two-parameter Weibull distribution, whose statistical properties are given as follows:

Mean: (MTTF or MTBF)

$$\mu_X = \beta\Gamma(1+1/\alpha) \qquad (3.19)$$

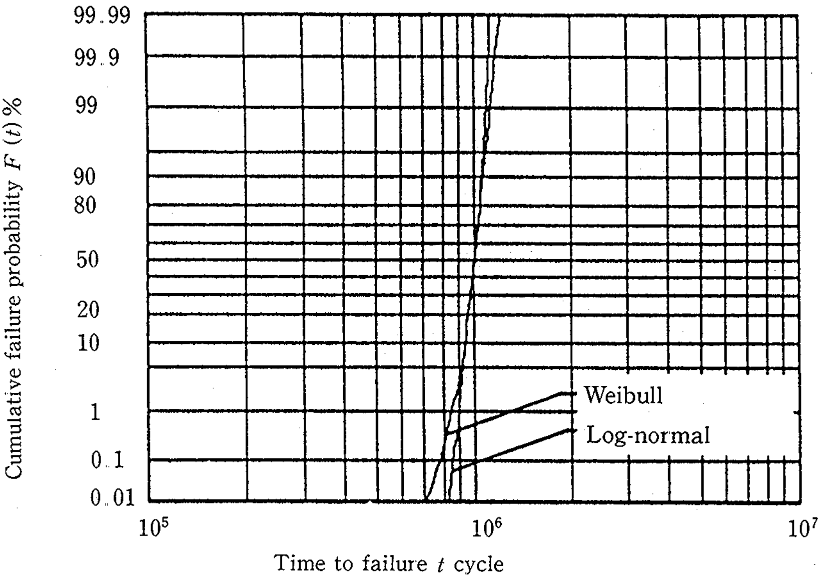
Variance:

$$\sigma_X^2 = \beta^2[\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)] \qquad (3.20)$$

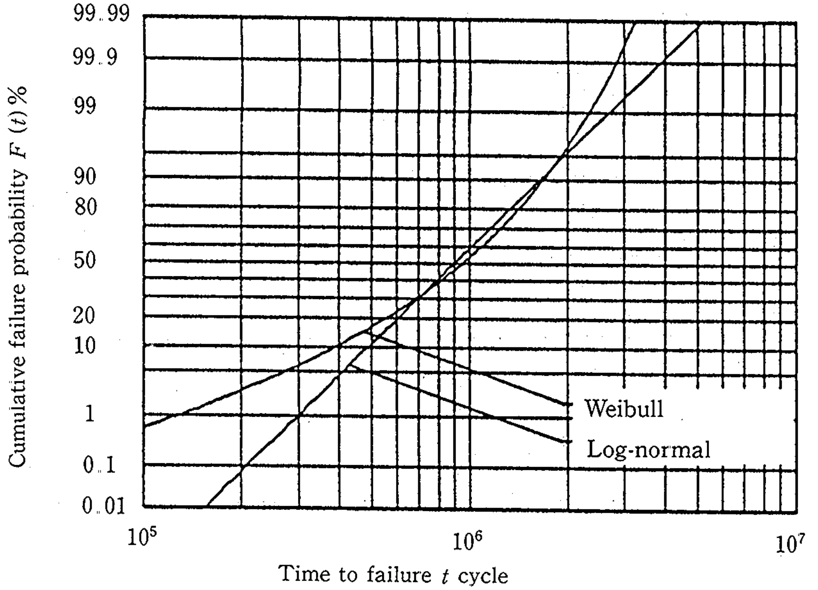
4 Wide Applicability of Weibull Distribution for Fatigue Life Scatters

4.1 Superiority of Weibull Distribution

As an interesting example of application of aforementioned stochastic models to the variability analysis of fatigue life T, a brief comparison is made between the Weibull distribution derived from the weakest link model and the log-normal distribution from the proportinal effect model with an

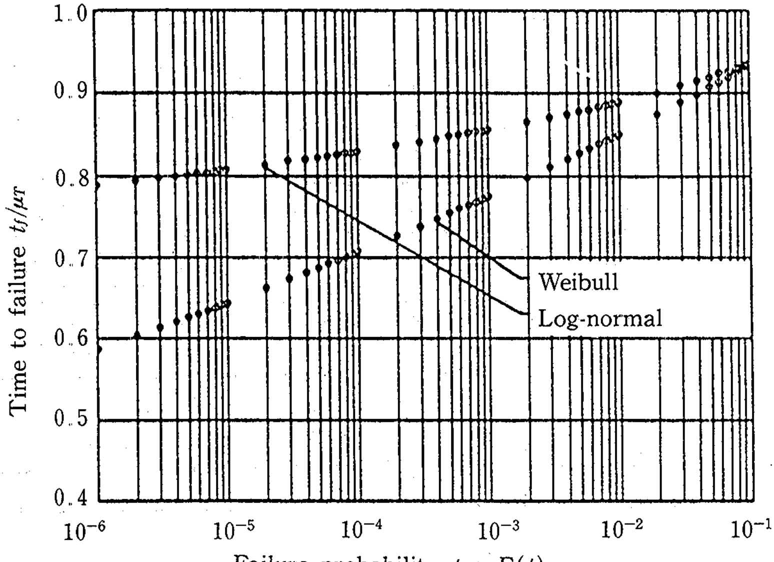


(a) In case that $\mu_T = 10^6$ and $V_T = 0.05$

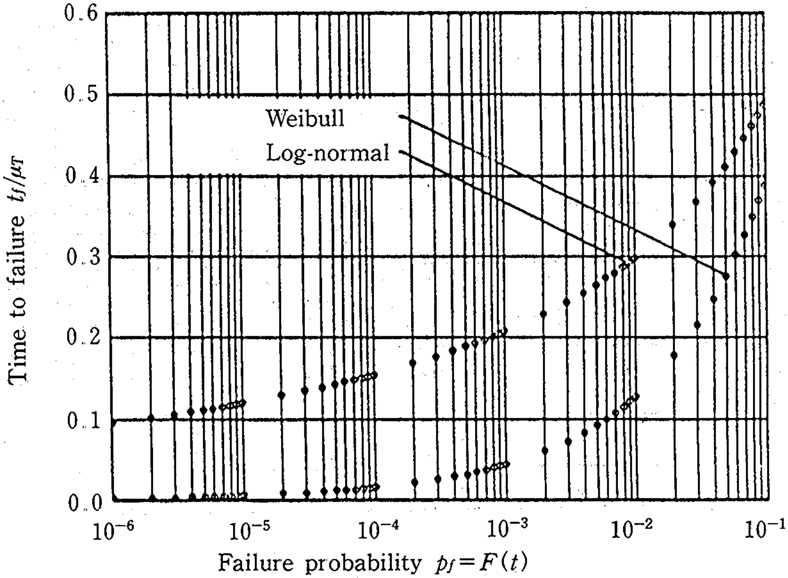


(b) In case that $\mu_T = 10^6$ and $V_T = 0.5$

Fig. 4. 1 Difference between shapes of cumulative distributions.



(a) In case that $\mu_T=10^6$ and $V_T=0.05$



(b) In case that $\mu_T=10^6$ and $V_T=0.5$

Fig. 4.2 Fatigue life percent points of both distributions.

emphasis on how these distributions behave in such a region of smaller failure probability as is usually the case in the practical design.

Fig. 4. 1(a) and (b) represent the difference between the shapes of cumulative distributions plotted on a log-normal probability paper in the case that mean life of each distribution is kept the same as $\mu_T = 10^6$ and coefficient of variation $V_T = 0.05$ and 0.5. As can be clearly seen, the Weibull distribution lies over the log-normal distribution in the region of smaller failure probability. This fact implies that the Weibull distribution gives a shorter percent point of life than the log-normal distribution.

Further, Fig. 4. 2(a) and (b) represent fatigue life percent points of both distributions as a function of fairly small failure probability. In every case, the Weibull distribution assumes a smaller value than that of the log-normal distribution, which means that the former lies in safer side than the latter. These results must be taken into account in the reliability-based fatigue-proof design.

4. 2 Weibull Distribution as Fatigue Life Distribution Model

As stated in the previous section, the well-known Weibull distribution with wide applicability is characterized by three parameters, that is, the shape parameter α, the scale parameter β and the location parameter γ. This is named after W. Weibull in Sweden who proposed this distribution [100]. In the Weibull distribution, the density function $f(t)$, the distribution function (sometimes called as unreliability function) $F(t)$, the reliability function $R(t)$ and the failure rate function $h(t)$ are given respectively as follows:

$$f(t) = \frac{\alpha}{\beta}\left(\frac{t-\gamma}{\beta}\right)^{\alpha-1} \exp\left\{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}\right\} \tag{4.1}$$

$$F(t) = 1-\exp\left\{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}\right\} \tag{4.2}$$

$$R(t) = 1 - F(t) = \exp\left\{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}\right\} \tag{4.3}$$

$$h(t) = f(t)/R(t) = \frac{\alpha}{\beta}\left(\frac{t-\gamma}{\beta}\right)^{\alpha-1} \tag{4.4}$$

in each equation, t assumes a value larger than or equal to γ; $t \geq \gamma$.

As can be seen from the shape of the failure rate function $h(t)$, three different patterns IFR, CFR and DFR of $h(t)$ can be produced by choosing a value of α such that $\alpha > 1$, $\alpha = 1$ and $0 < \alpha < 1$, respectively. This is why a Weibull distribution has wide applicability. In the above equations, β is sometimes called the characteristic life, and the scale parameter is defined by $t_0 = \beta^{\alpha}$. The physical meaning of the location parameter γ, whose interpretation shoud be of much consideration, may be regarded as the duration with no damage in degradation failure or the time to crack initiation in faitgue.

For the sake of ease of handling as well as of the fact that fracture might occur on the moment of the beginning of service, γ can be regarded as zero in what follows [105]. However, this treatment may not cause any lack of generality of discussion. In this case, Eqs. (4.1)-(4.4) reduce to the following forms:

$$f(t) = \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1} \exp\left\{-\left(\frac{t}{\beta}\right)^{\alpha}\right\} \tag{4.1$'$}$$

$$F(t) = 1 - \exp\left\{-\left(\frac{t}{\beta}\right)^{\alpha}\right\} \tag{4.2$'$}$$

$$R(t) = \exp\left\{-\left(\frac{t}{\beta}\right)^{\alpha}\right\} \tag{4.3$'$}$$

$$h(t) = \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1} \tag{4.4$'$}$$

where t assumes a value larger than or equal to zero; $t \geq 0$.

The shape of Weibull distribution varies as the shape parameter α changes. Fig. 4.3 indicates the variation of the density function in case of

$\gamma = 0$, from which we can see that the function corresponds to an exponential distribution in case of $\alpha = 1$, to a Rayleigh distribution in case of $\alpha = 2$, and nearly to a normal distribution in case of $\alpha = 3.2$. The shape of the distribution comes to stand sharply with increasing value of shape parameter α. The statistical properties of a two-parameter Weibull distribution defined by Eqs. (4. 1)′-(4. 4)′ are given as follows:

Mean (MTTF or MTBF):

$$E[T] = \beta \Gamma\left(1+\frac{1}{\alpha}\right) \tag{4.5}$$

Variance:

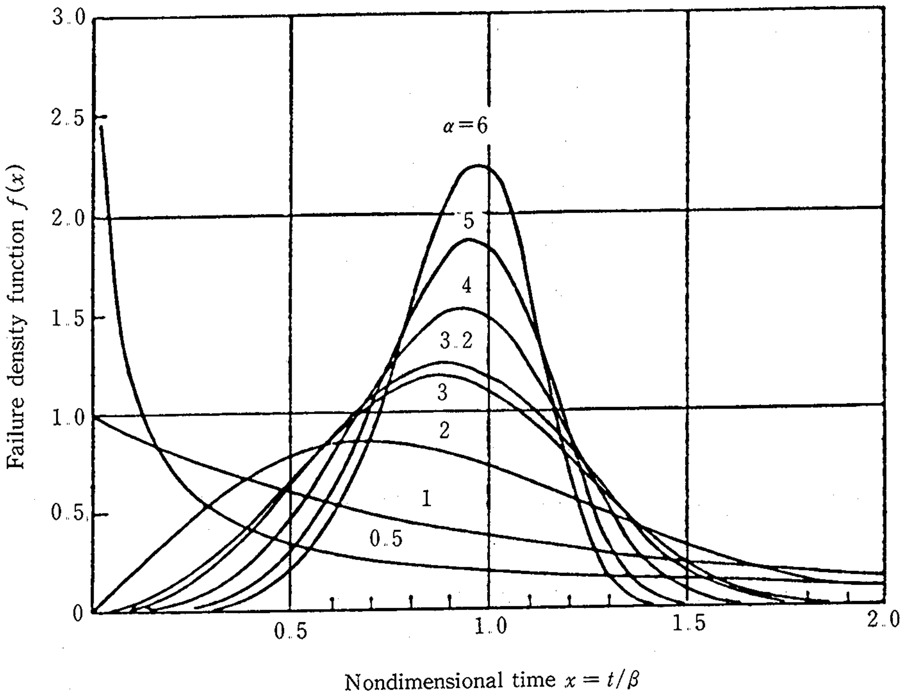


Fig. 4. 3 Effect of shape parameter α on Weibull distribution, where β represents a scale parameter.

$$\mathrm{Var}[T] = \beta^2\left[\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma^2\left(1+\frac{1}{\alpha}\right)\right] \tag{4.6}$$

Median:

$$t_{\mathrm{median}} = \beta(\ln 2)^{1/\alpha} \tag{4.7}$$

where T represents a random variable of the time to failure or fracture, and $\Gamma(\cdot)$ means a Γ function.

4.3 Weakest Link Model and Extreme Value Distribution

In this section, both the weakest link model and the extreme value distribution are briefly discussed. It is often the case in reliability problems that the Weibull distribution can predict the variability in the model of failure or fracture. This comes partly from the fact that the distribution belongs to one type of extreme value distributions.

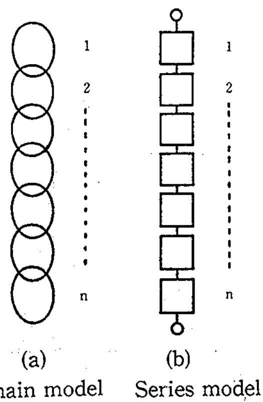


Fig. 4.4 Schematic representation of the weakest link, or the chain model and the series model.

With consideration of the chain model composed of mutually independent n links, each having the same strength or life distribution, as shown in Fig. 4.4, let us think of the strength or life of this chain model. Let $R_i(t)$ be the reliability of the i-th link at time t. Then the reliabilioy $\bar{R}(t)$ of the chain is given as follows:

$$\bar{R}(t) = \prod_{i=1}^{n} R_i(t) \tag{4.8}$$

In case that the reliability of each link is the same, that is, $R_i(t) = R(t)$, Eq. (4.8) reduces to the following.

$$\bar{R}(t) = \{R(t)\}^n \tag{4.9}$$

Such structural model as this is called the weakest link model or the chain model, and is also accordant with the series model in Fig. 4.4.

This concept can be extended to the determination either of the weakest strength or life distribution among n elements following the same distribution, or of the maximum stress distribution among n independent applications of the stress having a certain distribution. In the limit where n becomes infinity, the extreme value distribution is obtained as an asymptotic distribution. E. J. Gumbel [106] categorized this asymptotic distribution into three types; exponential type, Cauthy type and truncated type of distribution. The Weibull distribution belongs to the third type.

The importance of the concept of the extreme value distribution is mainly based on the following three facts:

(a) The life of an item can be deemd to be determined by the maximum size of the defects involved.

(b) The maximum of stress subjected to an item, for example, the maximum speed of wind will prescribe the life of the item.

(c) The annual maximum level of water in a river, the annual max-imum quantity of rainfall or the maximum magnitude of earthquake can be obtained with the aid of this concept.

5 Notion of Statistical Inference

Prior to the consideration of the statistical inference of parameter estimation, we briefly refer to the notion of the statistical inference [107],

[108].

In general, it is hardly the case that the probability distribution of the object of concern is determined in advance. In some cases, although the shape of the distribution function could be predicted based on the past experience, the distribution parameters such as mean, standard deviation, shape or scale parameter should be inferred based on samples of comparatively small size. The statistical inference is the procedure to be applied for this purpose.

As for the statistical inference, there are two categories: One is called the estimation to estimate unknown parameters which prescribe the population distribution. The other is the test to test whether or not estimated parameters of the population distribution are pertinent or the shape of distribution is acceptable. The present study mainly concerns with the former which is composed of the point estimation to determine the value of unknown parameter at a certain point of best feature, and interval estimation to determine the interval where the true value of unknown parameter exists with a certain prescribed confidence level.

5.1 Point Estimation

Even if we could assume the shape of the population distribution with a certain method, the distribution never comes to be fixed unless the values of parameters to characterize the distribution are determined. It is the point estimation that the value of parameter is estimated so as to have the best feature from a certain aspect. In this sense, what is the best feature becomes of vital importance.

Let $\Phi(T_1, T_2, \ldots, T_n)$ be an estimator of a certain parameter θ composed of a random sample of size n, $T_1, T_2, \ldots, T_n$. Then, the estimator Φ is preferably expected to have the following features (1) to (4):

(1) Consistency

This means that the following equation is satisfied for any small positive value ε. In other words, Φ converges to θ on a probabilistic basis.

$$\lim_{n\to\infty} P[|\Phi-\theta| \geq \varepsilon] = 0 \tag{5.1}$$

The estimator $\Phi(T_1, T_2, \ldots, T_n)$ which has this nature is called a consistent estimator.

(2) Unbiasedness

This means that the estimator $\Phi(T_1, T_2, \ldots, T_n)$ has no bias neither to upper side nor to lower side. That is,

$$E[\Phi(T_1, T_2, \ldots, T_n)] = \theta \tag{5.2}$$

The estimator $\Phi(T_1, T_2, \ldots, T_n)$ which has this nature is called an unbiased estimator. Here, $E[\Phi]-\theta$ is called bias and the estimator Φ whose bias is not zero is also called a biased estimator.

(3) Minimum variance

Even if the estimator satisfies the aforementioned unbiasedness in (2), the probability needs to be small that the estimator lies largely apart from the true value. To meet this requirement, the variance is desirable to be as small as possible.

Now suppose that $\Phi(T_1, T_2, \ldots, T_n)$ is the unbiased estimator of θ, then the variance is given as follows:

$$\begin{aligned} \mathrm{Var}[\Phi] &= E[\{\Phi-E[\Phi]\}^2] = E[\Phi^2]-(E[\Phi])^2 \\ &= E[\Phi^2]-\theta^2 \end{aligned} \tag{5.3}$$

where the value evidently changes according to the value of θ. Hence, if Φ^* exists which satisfies the following relationship for an arbitrary unbiased estimator Φ in every value of θ, this is named as the uniformly minimum variance unbiased estimator or UMV unbiased estimator and is also quite desirable.

$$\mathrm{Var}[\Phi] \geq \mathrm{Var}[\Phi^*] \tag{5.4}$$

By the way, assumed that a sample of size n, T_1, T_2, ..., T_n, are generated independently from the population with probability density function $f(t, \theta)$, the following Cramér-Rao inequality [109] is applicable to an arbitrary unbiased estimator Φ for θ. From this point, such an estimator whose variance always agrees with the limit of the right-hand side of the above inequality (this property is called efficiency) may be regarded as the most desirable.

$$\mathrm{Var}[\Phi] \geq -\left\{n\mathrm{E}\left[\frac{\partial^2}{\partial\theta_2}\log f(T_i,\ \theta)\right]\right\}^{-1}$$
$$(i=1,\ 2,\ \ldots,\ n) \qquad (5.5)$$

This is called an efficient estimator and is evidently a UMV unbiased estimator. However, in general, the UMV unbiased estimator, even if it exists, is not always an efficient estimator.

Well, in case that Φ is a biased estimator, the estimation of its accuracy is performed by the mean square error represented by $E[(\Phi-\theta)^2]$ in place of variance. In case of an unbiased estimator, it agrees with variance.

(4) Sufficiency

In the estimation of parameter θ, it is a sufficient estimator of θ that all information obtained from sample data is concentrated on the estimator and any other estimator cannot give more information than it. In other words, when a certain value of function $\Phi=\phi(T_1,\ T_2,\ \ldots,\ T_n)$ can be computed and this value can be sufficient for estimation in place of all sample values, the function Φ is called a sufficient estimator. This can also be defined by the following. That is, it is a sufficient estimator Φ when the conditional probability of T_1, T_2, ..., T_n given $\Phi=\phi$, that is, $\mathrm{P}[T_1,\ T_2,\ \ldots,\ T_n \mid \Phi=\phi]$, is always free from the parameter θ.

Undoubtedly, it is the most desirable for an estimator to have all of the abovementioned four features; consistency, unbiasedness, efficiency and sufficiency. However, it is often the case that some of these properties are not satisfied.

5.2 Interval Estimation

Although the point estimation has various desirable properties, the reliability placed on the value may not be so high, since estimated values may vary from sample to sample. For the obtained point estimate can be deemed to be close to the true parameter, if we set an interval with appropriate margin on both sides of the estimate, then the probability that the true parameter exists in this interval could be considerably high. This is the notion of the interval estimation which provides much higher reliability than the point estimation.

The interval estimation means to determine θ_L and θ_U as a function of a sample, $\theta_L(T_1, T_2, \ldots, T_n)$ and $\theta_U(T_1, T_2, \ldots, T_n)$, such that the probability that the interval (θ_L, θ_U) contains the true parameter θ is $(1-\alpha)$.

$$P[\theta_L \leq \theta \leq \theta_U] = 1-\alpha \equiv \gamma \tag{5.6}$$

When this interval (θ_L, θ_U) exists, this is called the interval with confidence coefficient $\gamma(=1-\alpha)$ or confidence interval of $100\gamma = 100(1-\alpha)\%$, and θ_L and θ_U are said the lower or the upper confidence limit, respectively. Such estimation with lower and upper confidence limits is called the two-sided confidence interval estimation, and in some cases, only one limit is of interest. This is called the one-sided confidence interval estimation. In this case,

$$P[\theta_L \leq \theta] = \gamma, \text{ or } P[\theta_U \geq \theta] = \gamma \tag{5.7}$$

Since the probability that the confidence interval (θ_L, θ_U) contains a true value of θ is $\gamma = 1-\alpha$, the probability that the true parameter lies outside of the interval is α. From this point of view, α is called the risk ratio. Either the risk ratio α or the confidence coefficient γ should be determined as the occasion may demand, but it is ordinary that $\gamma = 0.95(\alpha = 0.05)$ or $\gamma = 0.99(\alpha = 0.01)$ is adopted.

5.3 Basic Concept of Maximum Likelihood Estimation

As stated earlier, an estimator is desired to have consistency, unbi-asedness, efficiency and sufficiency. The maximum likelihood method is one of the practical methods to form a desirable estimator like this.

For the sake of simplicity, assumed that the random variable T of one population distribution has the probability density function $f(t\,;\,\theta)$ which depends only on one parameter θ, the joint probability density function of a sample of size n, $T_i(i = 1, 2, \ldots, n)$, extracted randomly from the population, is given as follows:

$$L = f(t_1, t_2, \ldots, t_n\,;\,\theta) = \prod_{i=1}^{n} f(t_i\,;\,\theta) \tag{5.8}$$

This is called the likelihood function in the sample point $(t_1, t_2, \ldots, t_n)$, whose values change obviously, depending on the parameter θ. At this time, it is natural to consider that the sample point $(t_1, t_2, \ldots, t_n)$ is most likely to realize when θ assumes $\hat{\theta}$ at which the likelihood function becomes the maximum. The maximum likelihood method is based on this concept. Therefore, $\hat{\theta}$ is usually given as a solution of the following equation:

$$dL/d\theta = 0 \tag{5.9}$$

$\hat{\theta}$ thus defined is called the maximum ldkelihood estimator (hereafter abbreviated as MLE). In case that the number of unknown parameters is r, similarly, the set of parameters of size $r(\theta_1, \theta_2, \ldots, \theta_r)$ may be obtained so as to maximize the following likelihood function.

$$L = \prod_{i=1}^{n} f(t_i\,;\,\theta_1, \theta_2, \ldots, \theta_r) \tag{5.10}$$

This will usually result in a set of solutions of the following system of equations derived by partially differentiating L with respect to θ_i:

$$\partial L/\partial\theta_i = 0, \ (i = 1, 2, \ldots, r) \tag{5.11}$$

It is based on the following reasons that the aforementioned MLE is really utilized quite often:

(1) In case that a sufficient estimator exists, MLE will become that estimator.

(2) MLE is not always an unbiased estimator. However, it is often the case that a simple modificatin can bring unbiasedness to MLE.

(3) In case that an efficient estimator of unknown parameter θ exists, MLE $\hat{\theta}$ becomes an efficient estimator of θ.

(4) In case of large sample size, MLE has the property to follow asymptotically the normal distribution. That is, in a sample of size n, $\sqrt{n}(\hat{\theta}-\theta)$ follows asymptotically the normal distribution with mean = 0 and variance $= n\{-n\mathrm{E}[\partial^2 \log f(t;\ \theta)/\partial\theta^2]\}^{-1}$. Hence, $\hat{\theta}$ becomes a consistent entimator of θ.

As can be seen clearly in the above expressions, MLE really has a lot of desirable properties.

6 Elimination of High-Time Outliers in a Sample from Weibull Population

In a Weibull model, when so-called high-time outlier (extremely large value) exists in a sample, distribution parameters cannot be estimated correctly. Let us think of two examples of Ex. 1 and Ex. 2 shown in Table 6.1 [10]. In both examples, most of data are the same, but the former contains one extremely large life. Conversely, the latter contains one very small life. Table 6. 1 represents point estimates of fatigue life at a certain failure probability based on the estimated shape and scale parameters, α and β, which are also point estimates with the aid of MLE discussed later. According to this Table, Ex. 1 shows extremely large scatter and Ex. 2 does not. That is, in a Weibull model, estimates are much affected by high -time outliers. On the other hand, they are not influenced so much by sample values which are extremely small. In reference to this fact, we should not consider that a Weibull model is not applicable to fatigue life distribution because an estimate is easily affected by high-time outlier. In the estimation, it is better to remove such small quantities of data as those extracted from the different population with some reasons. One of such eliminating procedures is an estimation by MLE-censored, whose example is illustrated in Table 6. 2. In Table 6. 2, Ex. 2 is the case that two high-time outliers are added to Ex. 1 which contains no outlier, and reversely Ex. 3 is the case that two low-time outliers are added. In Ex. 2, when the estimation of the shape parameter is performed by replacing each outlier by the sample value immediately preceding it, we can observe that the estimate of shape parameter changes largely in the second censoring of outlier and after that no remarkable change is observed. This is why the third is chosen as the estimate. On the other hand, in Ex. 3 the first estimate is accepted because the censoring gives no considerable change to an estimate.

Table 6. 1 Simulated examples to illustrate effects of isolated long-life specimens.

Example	Fatigue life(cycles)	Estimates of:		Point estimates of life (cycles) at some failure probabilities
		Characteristic life β(cycles)	Weibull shape α	
1	42000 45000 48000 52000 55000 60000 500000	108040	0.887	50% = 71480 10% = 8554 5% = 3800 1% = 605 0.1% = 45 0.01% = 3
2	5000 42000 45000 48000 52000 55000 60000	47680	2.541	50% = 41273 10% = 19662 5% = 14811 1% = 7798 0.1% = 3145 0.01% = 1270

Table 6. 2 Simulated examples to illustrate censoring procedure[10].

	Example 1		Example 2		Example 3	
	Fatigue life (cycles)	Weibull shape $\hat{\alpha}$ ($\hat{\lambda}$)	Fatigue life (cycles)	Weibull shape $\hat{\alpha}$ ($\hat{\lambda}$)	Fatigue life (cycles)	Weibull shape $\hat{\alpha}$ ($\hat{\lambda}$)
Original estimate	42000 45000 48000 52000 55000 60000	9.07 (0.11)	42000 45000 48000 52000 55000 60000 400000 500000	0.95 (1.05)	4000 5000 42000 45000 48000 52000 55000 60000	1.63 (0.61)
Second estimate			42000 45000 48000 52000 55000 60000 400000 400000→	1.05 (0.95)	4000 5000 42000 45000 48000 52000 55000 55000→	1.50 (0.67)
Third estimate			42000 45000 48000 52000 55000 60000 60000→ 60000→	9.07 (0.11)	4000 5000 42000 45000 48000 52000 52000→ 52000→	1.35 (0.74)
Fourth estimate			42000 45000 48000 52000 55000 55000→ 55000→ 55000→	11.70 (0.085)	4000 5000 42000 45000 48000 48000→ 48000→ 48000→	1.20 (0.83)
Answer	Because original was O. K., no attempt was made to censore.		Third estimate	9.07	Original estimate	1.63

7 MLE of Parameters in a Two-Parameter Weibull Model

7.1 In case that both parameters are unknown

Assumed that fatigue life T is a random variable which follows a two-parameter Weibull distribution, parameters to describe this distribution are the shape parameter α and the scale parameter β. The probability density function $f(t)$ and the distribution function $F(t)$ are given as follows:

$$f(t)=\frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1}\exp\left\{-\left(\frac{t}{\beta}\right)^{\alpha}\right\} \tag{7.1}$$

$$F(t)=1-\exp\left\{-\left(\frac{t}{\beta}\right)^{\alpha}\right\} \tag{7.2}$$

Let us suppose that fatigue life is obtained by independent fatigue tests on n test pieces and that each follows the same Weibull life distribution. The outcome in each test is either

(a) the time to actual failure of test piece (fatigue life), T

or

(b) the random time to terminate test for any reason other than fracture of test piece (censoring time), Z

where T and Z should be treated as random variables. The symbolic representation of the outcome in each test is as follows:

$$[T=t]\cap[T\leq Z] \text{ or } [Z=z]\cap[T>Z] \tag{7.3}$$

where t or z is one realization of T or Z, obtained by a test of each time.

At this point, assume that k specimens out of $n(k\leq n)$ are tested to failure and the test on remaining $(n-k)$ specimens are terminated by the reason other than the failure of the specimen.

Consequently, from the former, k outcomes of fatigue life $T(t_1, t_2, \ldots, t_k)$ are obtained and, from the latter, those of censoring time Z of size $(n-k)$, $(z_{k+1}, z_{k+2}, \ldots, z_n)$, are also gained, where suffix dedicated to each outcome is for convenient purpose and has no special meaning to show the order of size. Since the outcome of each test is shown in Eq. (7. 3), the event that the set of n outcomes $(t_1, t_2, \ldots, t_k, z_{k+1}, z_{k+2}, \ldots, z_n)$ will occur is represented, based on the concept of the joint event, as follows:

$$\bigcap_{i=1}^{k} \{[T_i \leq Z_i] \cap [T_i = t_i]\} \bigcap_{j=k+1}^{n} \{[T_j > Z_j] \cap [Z_j = z_j]\} \qquad (7.4)$$

where symbol $\bigcap_{i=1}^{k}\{E_i\}$ denotes the joint event $E_1 \cap E_2 \cap \cdots \cap E_k$. Therefore, the likelihood L of such event will be represented as follows:

$$L = C\prod_{i=1}^{k} f(t_i) \prod_{j=k+1}^{n} \{1-F(z_j)\} \qquad (7.5)$$

where C is a constant value independent of parameters α and β in case that the set of complete outcomes $(T_1 = t_1,\ T_2 = t_2, \ldots, T_k = t_k)$ of T is given.

Since the population density function $f(t)$ and the distribution function $F(t)$ are given in Eq. (7. 1) and (7. 2) respectively, the following can be derived by substituting these into Eq. (7. 5):

$$L = C \prod_{i=1}^{k} \left[\frac{\alpha}{\beta}\left(\frac{t_i}{\beta}\right)^{\alpha-1} \exp\left\{-\left(\frac{t_i}{\beta}\right)^{\alpha}\right\}\right] \prod_{j=k+1}^{n} \left[\exp\left\{-\left(\frac{z_j}{\beta}\right)^{\alpha}\right\}\right] \qquad (7.6)$$

The log-likelihood is given by taking the logarithm of Eq. (7. 6) as

$$\ln L = \ln C + \sum_{i=1}^{k} \left\{\ln\left(\frac{\alpha}{\beta}\right) + (\alpha-1)\ln\left(\frac{t_i}{\beta}\right) - \left(\frac{t_i}{\beta}\right)^{\alpha}\right\} - \sum_{j=k+1}^{n} \left(\frac{z_j}{\beta}\right)^{\alpha} \qquad (7.7)$$

Attentive to the fact that C is independent of parameters, the MLE's of the Weibull shape parameter α and scale parameter β are obtained, from aforementioned Eq. (5. 11), as the solution of the following system of equations:

$$\left.\begin{aligned} \frac{\partial}{\partial\alpha}\ln L &= \frac{k}{\alpha} + \sum_{i=1}^{k} \ln\left(\frac{t_i}{\beta}\right) - \sum_{j=1}^{k} \left(\frac{t_i}{\beta}\right)^{\alpha} \ln\left(\frac{t_i}{\beta}\right) \\ &\quad - \sum_{j=k+1}^{n} \left(\frac{z_j}{\beta}\right)^{\alpha} \ln\left(\frac{z_j}{\beta}\right) = 0 \\ \frac{\partial}{\partial\beta}\ln L &= -k\frac{\alpha}{\beta} + \frac{\alpha}{\beta}\left\{\sum_{i=1}^{k}\left(\frac{t_i}{\beta}\right)^{\alpha} + \sum_{j=k+1}^{n}\left(\frac{z_j}{\beta}\right)^{\alpha}\right\} = 0 \end{aligned}\right\} \qquad (7.8)$$

Suppose that the MLE's are made $\hat{\alpha}$ and $\breve{\beta}$ respectively in case that both parameters α and β are unknown, they can be obtained as the solution of the following simultaneous equations [110], which are derived by transforming Eq. (7. 8):

$$\frac{1}{k}\left\{\sum_{i=1}^{k}\left(\frac{t_i}{\breve{\beta}}\right)^{\hat{\alpha}}+\sum_{j=k+1}^{n}\left(\frac{z_j}{\breve{\beta}}\right)^{\hat{\alpha}}\right\}=1 \tag{7. 9}$$

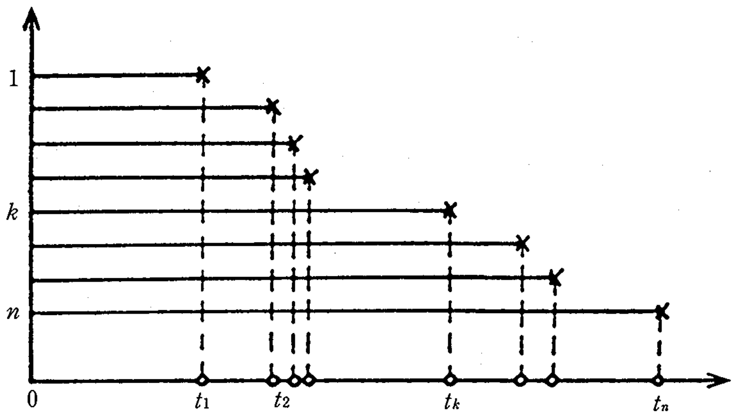
$$\frac{k}{\hat{\alpha}}=\sum_{i=1}^{k}\left(\frac{t_i}{\breve{\beta}}\right)^{\hat{\alpha}}\ln\left(\frac{t_i}{\breve{\beta}}\right)+\sum_{j=k+1}^{n}\left(\frac{z_j}{\breve{\beta}}\right)^{\hat{\alpha}}\ln\left(\frac{z_j}{\breve{\beta}}\right)^{\hat{\alpha}}-\sum_{i=1}^{k}\ln\left(\frac{t_i}{\breve{\beta}}\right) \tag{7. 10}$$

By the way, in the actual fatigue test, one of the following test methods (a) to (c) will generally be adopted.

(a) Uncensored testing plan

As illustrated in Fig. 7. 1, the tests are performed until all specimens of size n will fail. When they are tested all at the same time, the observed data are obtained in order of magnitude as follows:

$t_1 \leq t_2 \leq \ldots \leq t_n$



n complete observations

Fig. 7. 1 Uncensored testing plan

Data of size n, ordered in this fashion, are called order statistics of size n and a sample obtained by this testing plan is called uncensored sample. This method is applicable to the case that a certain estimation is made by use of the entire sample of size n, without any modification, if data group consisting of data of size n are gained. This type of sample is sometimes called complete sample because the whole members of a sample are used.

(b) Fixed time testing plan

As represnted in Fig. 7. 2, the test is terminated after a certain lapse of time. When the censoring time τ locates as shown in the figure, failure times of t_1, t_2, ..., t_k are obtained. On the other hand, there is only such information that each of t_{k+1}, t_{k+2}, ..., t_n is larger than τ. This kind of sample is termed as the type I censored sample.

(c) Fixed number testing plan

As shown in Fig. 7. 3, the test is terminated when a prescribed number of specimens $k(1 \leq k \leq n)$ fails. In this case, the values of t_1, t_2, ..., t_k are known, but as for t_{k+1}, t_{k+2}, ..., t_n, there exists only such information that each of them is not smaller than t_k. This kind of sample is called the type II censored sample and k is said the censoring number.

As stated earlier, the censoring procedure plays an important role in case of Weibull distribution where parameter estimates are largely affected by high-time outliers. The estimation procedure in Table 6.2 is the method that the estimation of the shape parameter α is performed based on a censored sample such that larger values than the k-th order statistic are replaced by the k-th order statistic. Therefore, this sample is the same as the type II censored sample.

Since, in the parameter estimation, the available data are generally given in a form of order statistics and the censoring procedure to censor at the k-th order statistic is usually applied, it will be convenient to describe

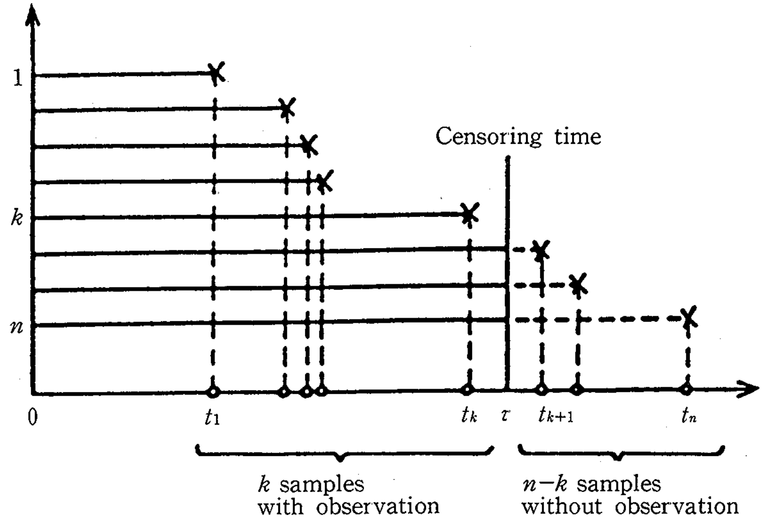


Fig. 7. 2 Fixed time testing plan

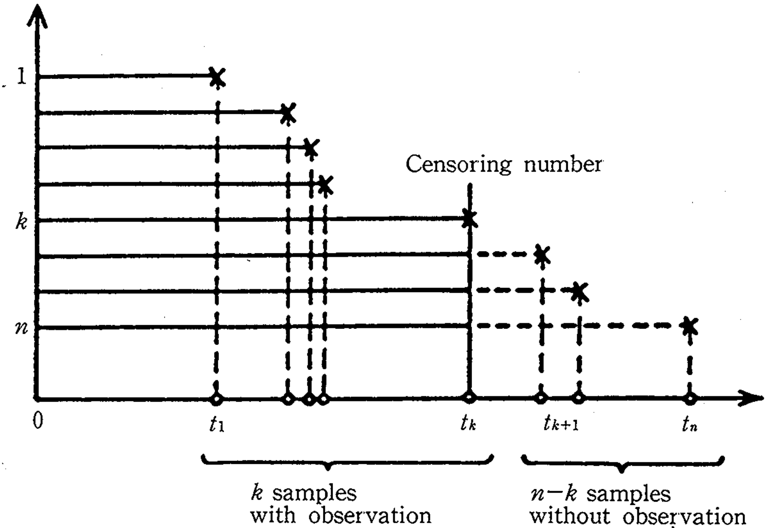


Fig. 7. 3 Fixed number testing plan

the MLE corresponding to this case. Here, suppose that n data exist in the data group, given as

$$t^{(1)}, t^{(2)}, \ldots, t^{(n)}$$

Then the order statistics arranged in ascending order of magnitude are represented as follows:

$$t_1 \leq t_2 \leq \ldots \leq t_n$$

At first, the MLE's (so-called MLE-uncensored) of α and β are discussed based on a complete sample of size n. In this case, the actual outcome of the censoring time Z is naught and hence, in Eq. (7.9) and (7.10), we may put $k=n$ and $z_j=0$. Therefore, in case that both parameters are unknown, the MLE-uncensored of α and β are obtained as $\hat{\alpha}$ and $\breve{\beta}$ which satisfy the following equations:

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{t_i}{\breve{\beta}}\right)^{\hat{\alpha}}=1 \tag{7.11}$$

$$\frac{n}{\hat{\alpha}}=\sum_{i=1}^{n}\left\{\left(\frac{t_i}{\breve{\beta}}\right)^{\hat{\alpha}}-1\right\}\ln\left(\frac{t_i}{\breve{\beta}}\right) \tag{7.12}$$

Second, the MLE's (so-called MLE-censored) of α and β are discussed based on a type II censored sample such that all the order statistics greater than the k-th value $(1 \leq k \leq n)$ are replaced by t_k.

$$t_1,\ t_2,\ \ldots,\ t_k=t_{k+1}=\ldots=t_n \tag{7.13}$$

Replacing $z_j \equiv t_k (j=k+1,\ k+2,\ \ldots,\ n)$ in Eq. (7.9) and (7.10), it follows that:

$$\frac{1}{k}\left\{\sum_{i=1}^{k}\left(\frac{t_i}{\breve{\beta}}\right)^{\hat{\alpha}}+(n-k)\left(\frac{t_i}{\breve{\beta}}\right)^{\hat{\alpha}}\right\}=1 \tag{7.14}$$

$$\frac{k}{\hat{\alpha}}=\sum_{i=1}^{k}\left[\left\{\ln\left(\frac{t_i}{\breve{\beta}}\right)\right\}\left\{\left(\frac{t_i}{\breve{\beta}}\right)^{\hat{\alpha}}-1\right\}\right]+(n-k)\left(\frac{t_k}{\breve{\beta}}\right)^{\hat{\alpha}}\ln\left(\frac{t_k}{\breve{\beta}}\right)^{\hat{\alpha}} \tag{7.15}$$

The MLE-censored of α and β are obtained as $\hat{\alpha}$ and $\breve{\beta}$ which satisfy the above equations, Eqs. (7.14) and (7.15).

The MLE's of α and β are obtained by the solution of Eqs. (7.9) and (7.

10), Eqs. (7. 11) and (7. 12), or Eqs. (7. 14) and (7. 15), respectively. However, the solution of these equations cannot be obtained in a closed form and therefore, it is not easy and is necessary to make use of iterative procedure with the aid of a computer. In the estimation of the shape parameter α, when the MLE of the reciprocal shape parameter $\lambda = 1/\alpha$ is considered, this value is obtained in the following way. That is, in case of MLE-uncensored where a complete sample is used, from Eqs. (7. 11) and (7. 12),

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{t_i}{\breve{\beta}}\right)^{1/\hat{\lambda}} = 1 \qquad (7.16)$$

$$n\hat{\lambda} = \sum_{i=1}^{n}\left\{\left(\frac{t_i}{\breve{\beta}}\right)^{1/\hat{\lambda}} - 1\right\}\ln\left(\frac{t_i}{\breve{\beta}}\right) \qquad (7.17)$$

In case of MLE-censored where a censored sample is used, from Eqs. (7. 14) and (7. 15),

$$\frac{1}{k}\left\{\sum_{i=1}^{k}\left(\frac{t_i}{\breve{\beta}}\right)^{1/\hat{\lambda}} + (n-k)\left(\frac{t_k}{\breve{\beta}}\right)^{1/\hat{\lambda}}\right\} = 1 \qquad (7.18)$$

$$k\hat{\lambda} = \sum_{i=1}^{k}\left[\left\{\ln\left(\frac{t_i}{\breve{\beta}}\right)\right\}\left\{\left(\frac{t_i}{\breve{\beta}}\right)^{1/\hat{\lambda}} - 1\right\}\right] + (n-k)\left(\frac{t_k}{\breve{\beta}}\right)^{1/\hat{\lambda}}\ln\left(\frac{t_k}{\breve{\beta}}\right)^{1/\hat{\lambda}} \qquad (7.19)$$

By solving the above equations with respect to $\breve{\beta}$ and $\hat{\lambda}$, $\hat{\lambda}$ gives an answer.

7. 2 In case that the shape parameter α is known

There may be the case that the shape parameter α is known for any reason. If the MLE of β corresponding to this case is expressed by $\hat{\beta}$, this is obtained by

$$\left.\begin{aligned}\hat{\beta} &= \left\{\frac{1}{k}\left(\sum_{i=1}^{k} t_i^{\alpha} + \sum_{j=k+1}^{n} z_j^{\alpha}\right)\right\}^{1/\alpha} \\ &= \left\{\frac{1}{k}\left(\sum_{i=1}^{k} t_i^{1/\lambda} + \sum_{j=k+1}^{n} z_j^{1/\lambda}\right)\right\}\end{aligned}\right\} \qquad (7.20)$$

On the other hand, the MLE-censored of β, $\hat{\beta}$, by use of such censoring procedure as shown in Eq. (7. 21), is obtained by transforming Eqs. (7. 14) and (7. 15) as follows:

$$Z_{k+1} = Z_{k+2} = \dots = Z_n \equiv T_k \tag{7.21}$$

where T_k is the k-th order statistic.

$$\begin{aligned} \hat{\beta} &= \left\{ \frac{1}{k} \left(\sum_{i=1}^{k} t_i^{\alpha} + (n-k) t_k^{\alpha} \right) \right\}^{1/\alpha} \\ &= \left\{ \frac{1}{k} \left(\sum_{i=1}^{k} t_i^{1/\lambda} + (n-k) t_k^{1/\lambda} \right) \right\}^{\lambda} \end{aligned} \tag{7.22}$$

In case of MLE-uncensored, it follows from Eq. (7.11) and (7.16) that

$$\begin{aligned} \hat{\beta} &= \left(\frac{1}{n} \sum_{i=1}^{n} t_i^{\alpha} \right)^{1/\alpha} \\ &= \left(\frac{1}{n} \sum_{i=1}^{n} t_i^{1/\lambda} \right)^{\lambda} \end{aligned} \tag{7.23}$$

Therefore, in case that α (or λ) is known, the maximum likelihood eatimate of Weibull scale parameter β, based on the result of the fullscale fatigue test, is obtained by Eq. (7.20), (7.22) or (7.23).

8 Statistical Properties of MLE'S of Weibull Parameters

From the discussion in the preceding section, the Weibull MLE's $\hat{\alpha}$ and $\breve{\beta}$, in case that both parameters are unknown, or $\hat{\beta}$, in case that the shape parameter is known, can be obtained. Hence, the first objective of the reliability-based design seems to be accomplished. However, if the distributions of $\hat{\alpha}$ and $\breve{\beta}$ or that of $\hat{\beta}$ can be predicted in advance, which represents how they distribute around the corresponding true values of α and β in relation to the sample size n and the censoring number k, the validation of the results of the actual estimation as well as the propriety of the sample size can be evaluated. This is much helpful, convenient and useful in a practical sense.

In the prediction of the distributions of $\hat{\alpha}$ and $\breve{\beta}$, the notion of parameter-free statistics independent of the true values α and β is desirable in order to give generality to the discussion. For this purpose, the following transformations are introduced:

$$Y_i = \left(\frac{T_i}{\beta}\right)^{\alpha},\ (i = 1,\ 2,\ \ldots,\ n) \tag{8.1}$$

$$U = \hat{\alpha}/\alpha \tag{8.2}$$

$$V = (\breve{\beta}/\beta)^{\hat{\alpha}} \tag{8.3}$$

where T_i is the i-th order statistic of size n.

Rewriting Eqs. (7.14) and (7.15) by use of Eqs. (8.1)~(8.3) reduces to:

$$V = \frac{1}{k}\left\{\sum_{i=1}^{k} y_i^U + (n-k) y_k^U\right\} \tag{8.4}$$

$$\frac{\sum_{i=1}^{k} y_i^U \ln y_i + (n-k) y_k^U \ln y_k}{\sum_{i=1}^{k} y_i^U + (n-k) y_k^U} - \frac{1}{U} = \frac{1}{k}\sum_{i=1}^{k} \ln y_i \tag{8.5}$$

By the way, it is evident that $y_i (i = 1,\ 2,\ \ldots,\ k)$ obtained by the transformation defined by Eq. (8.1) for fatigue life t_i $(i = 1, 2, \ldots, k)$ of a two-parameter Weibull type, follows the exponential distribution with mean of unity, independent of the Weibull two parameters α and β. Therefore, both U and V defined by Eqs. (8.2) and (8.3) are proved to be parameter-free statistics from the parameters α and β. In case of MLE-uncensored, similarly, both U and V are briefly shown to be parameter-free statistics from α and β, and the following equations hold:

$$V = \frac{1}{n}\sum_{i=1}^{n} y_i^U \tag{8.6}$$

$$\frac{\sum_{i=1}^{n} y_i^U \ln y_i}{\sum_{i=1}^{n} y_i^U} - \frac{1}{U} = \frac{1}{n}\sum_{i=1}^{n} \ln y_i \tag{8.7}$$

From the above discussion, the empirical distributions of U and V can be made clear by computer simulation with the aid of Monte Carlo technique. The detailed description of the simulation method is omitted here for lack of space. In what follows, the empirical distributions of U, V and W are briefly discussed.

Fig. 8.1 represents the distribution of $1/U = \alpha/\hat{\alpha} = \hat{\lambda}/\lambda$, reciprocal of

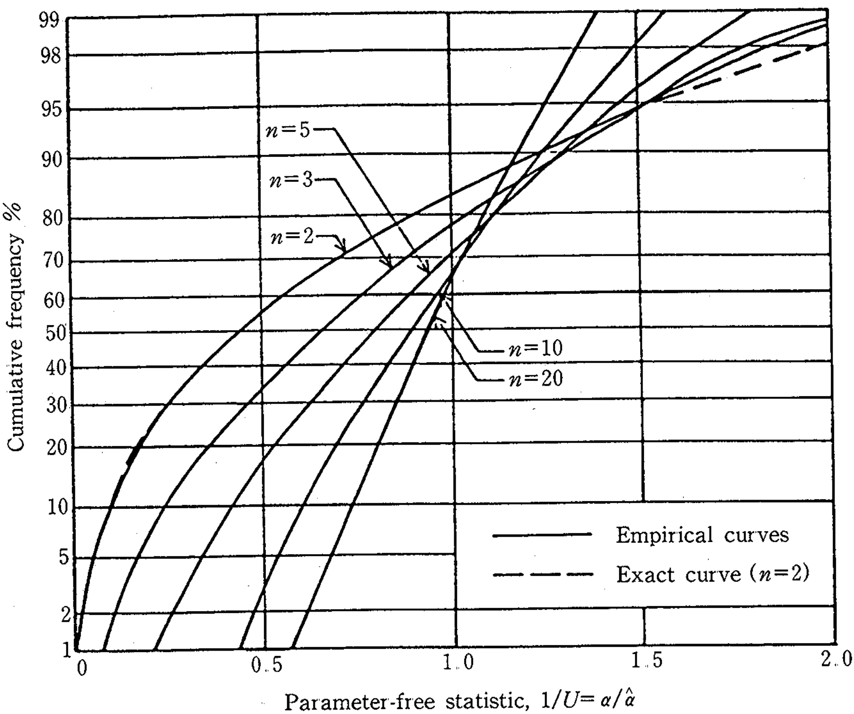


Fig. 8.1 Empirical distribution of the MLE of the Weibull shape parameter α for complete samples. (Note: $\hat{\alpha}$ is the MLE of the true Weibull shape α)

$U = \hat{\alpha}/\alpha$, obtained by computer simulation and plotted on a normal probability paper for each sample size n. For the sake of simplicity, only the case is shown here for complete samples without censoring procedure. The empirical distribution in the figure reprenents cumulative frequency of two thousand estimates of $1/U$, arranged in order of magnitude, obtained by utilizing those two thousand data groups numerically generated by a computer, each consisting of a sample of a given size of exponential random numbers with mean = 1. As shown clearly in the figure, with increasing sample size, the distribution of $1/U$ approaches a straight line whose slope

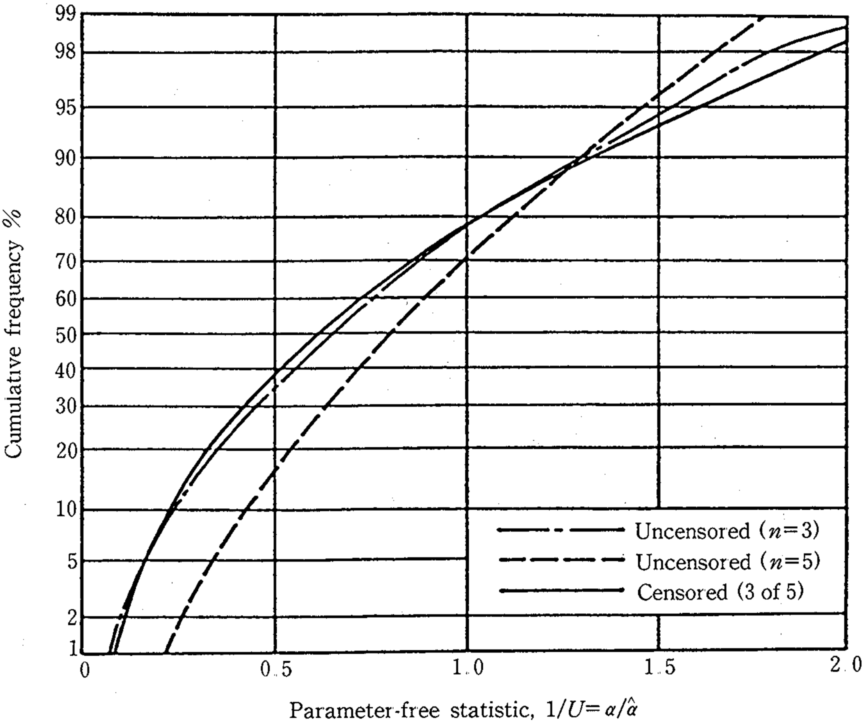


Fig. 8. 2 Comparison of the empirical distribution of the MLE of the Weibull shape parameter α between complete samples and censored one.

becomes gradually steeper. This means that the distribution of $1/U$ approaches a normal distribution with increasing sample size and that the probability that an estimate lies close to the true value becomes large since a steeper slope corresponds to smaller variance. Compared with the distribution of $V = (\breve{\beta}/\beta)^{\hat{\alpha}}$ discussed later, a considerably accurate estimation can be performed even for a small sample size. In case of $n = 2$, the distribution of $1/U$ is derived theoretically [111]. The estimated results by Monte Carlo simulation technique for $n = 2$ show a good agreement with the corresponding theoretical one represented by the dotted line. This fact can be the evidence of the validity of the present simulation technique.

Fig. 8. 2 gives an example of the simulated result of the effect of the censoring number on the estimation of α. Comparison between the censoring at the 3rd of 5 and complete sample of size 3 or 5 is illustrated. From this figure, the MLE of the shape parameter α is not largely affcted by the censoring and is estimated similarly for the small sample size [10].

Fig. 8. 3 represents how the sample size influences on the estimation accuracy in case of estimating the Weibull shape α by the MLE-unconsored $\hat{\alpha}$. The variance is usually a measure for estimation accuracy. However, if an estimattor has no unbiasedness, this is not a suitable measure and the expected loss or mean square error should be used in place of it. In this respect, both are represented in the figure. In case of small sample size, the difference between the expected loss and the variance becomes large,

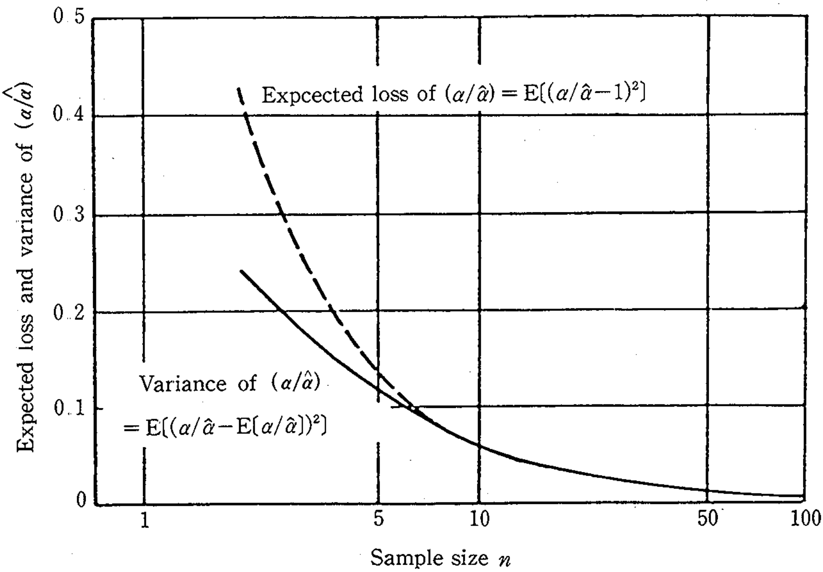


Fig. 8. 3 A measure of sampling error of the MLE of the Weibull shape from complete samples.

and the estimated error becomes also large.

Though the unbiasedness is a preferable feature for an estimator as stated earlier, $\hat{\alpha}$ discussed in this section is not always an unbiased estimator. For example, in case of sample size $n = 20$, $\hat{\alpha}$ seems to represent a normal distribution as shown in Fig. 8.1, but actually it has a little bias because the value at 50% of cumulative frequency is a litte less than the true value of unity. From this point of view, both the bias factor B_n and the variance factor Q_n of the MLE of the reciprocal shape parameter $\hat{\lambda} = 1/\hat{\alpha}$, represented by Eq. (8.8), are introduced and simulated for each sample size as in Table 8.1.

$$\mathrm{E}[B_n\hat{\lambda}] = \lambda, \text{ and } \mathrm{Var}[B_n\hat{\lambda}] = Q_n\lambda^2 \tag{8.8}$$

The reciprocal shape parameter λ is first estimated by MLE ($\hat{\lambda}$) for each data group. Then, according to the sample size, a modification is introduced as

$$\hat{\hat{\lambda}} = B_n\hat{\lambda} \tag{8.9}$$

$\hat{\hat{\lambda}}$ thus obtained now becomes an unbiased estimator. Assumed that there are m data groups, the best unbiased estimator $\hat{\hat{\lambda}}'$ throughout the whole groups can be given as follows:

$$\hat{\hat{\lambda}}' = \frac{\sum_{j=1}^{m} \hat{\hat{\lambda}}_j/Q_{nj}}{\sum_{j=1}^{m} 1/Q_{nj}} \tag{8.10}$$

with its variance defined as

$$\mathrm{Var}[\hat{\hat{\lambda}}'] = \lambda^2/\sum_{j=1}^{m}(1/Q_{nj}) \tag{8.11}$$

where $\hat{\hat{\lambda}}_j = B_{n_j}\hat{\lambda}_j$ is an unbiased estimator obtained by multiplying the MLE, $\hat{\lambda}$, estimated from the j-th data group by the bias factor B_{n_j} in Table 8.1, and Q_{n_j} is the variance fator corresponding to the sample size in the j-th data group.

Next, in case that the shape parameter α is unknown, the sampling

Table 8. 1 Bias and variance factor of Weibull MLE of reciprocal shape parameter $\hat{\lambda}$.

Complete sample size n	Bias factor B_n	Variance factor Q_n
2	1.73	0.71
3	1.37	0.35
4	1.25	0.22
5	1.187	0.164
10	1.088	0.073
20	1.047	0.033
∞	1	0

distribution of the aforementioned statistic defined by Eq. (8. 3) provides how the MLE of β, $\breve{\beta}$, distributes around the true value of β according to the sample size.

$$V = (\breve{\beta}/\beta)^{\hat{\alpha}} = (\breve{\beta}/\beta)^{1/\hat{\lambda}} \qquad (8.12)$$

Similarly to the empirical distribution of $1/U = \alpha/\hat{\alpha}$, some examples of the empirical distribution of the statistic V are represented in Fig. 8. 4 for complete samples, obtained by computer simulation by use of Eqs. (8. 6) and (8. 7). From this figure, in case of small sample size $n = 2 \sim 5$, a large scatter is observed in the distribution of V, which means that the probability that an estimate $\breve{\beta}$ is different from the true value is large, in the estimation of scale parameter β from data of sample size $n = 2 \sim 5$ when the Weibull shape α is assumed to be unknown. Therefore, a large value of the safety factor should be chosen in the design. Like the previous case of α, it asymptotically approaches a normal distribution when sample size becomes large. Fig. 8. 5 represents the effect of the censoring procedure on the results of estimation. In this case, the probability that β approaches the true value becomes large by applying the censoring procedure.

Finally, the discussion is made on how the MLE $\hat{\beta}$ distributes around the true value β when α is assumed to be known. Let us consider W

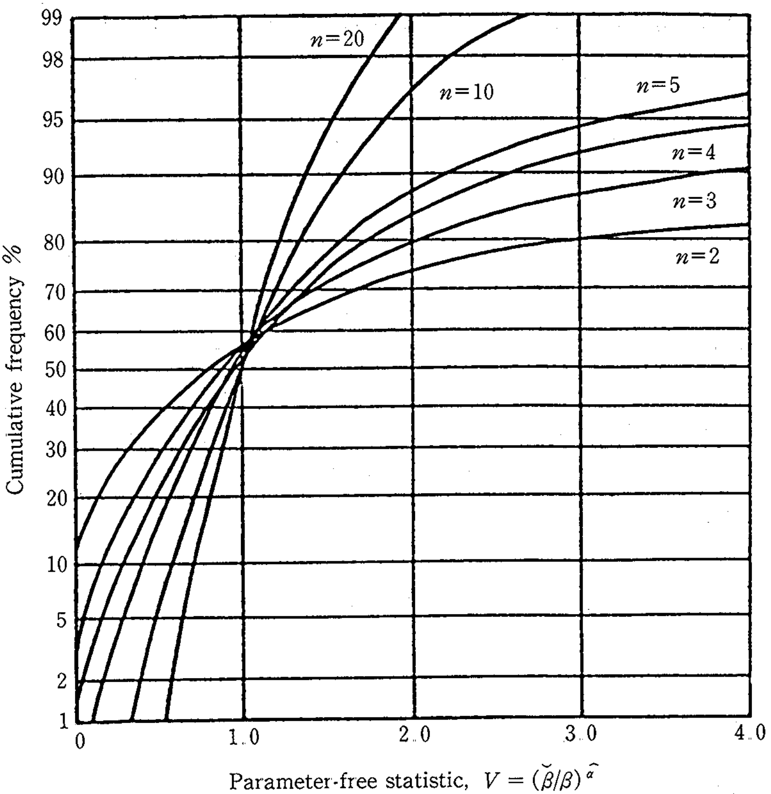


Fig. 8. 4 Empirical distribution of the MLE of the Weibull scale parameter β for complete samples of size n = 2, 3, 4, 5, 10 and 20 when Weibull shape is assumed to be unknown.

defined as

$$W = (\hat{\beta}/\beta)^{\alpha} \qquad (8.13)$$

The consideration only of the case of a censored sample as in Eq. (7. 22) suffices the discussion since it can easily be extended to the case of uncensored complete samples, by setting $k = n$. $\hat{\beta}$ with uncensored data is given by Eq. (7. 23). When we introduce the variate Y_i, obtained by transforming fatigue life T_i by Eq. (8. 1), W defined by Eq. (8. 13) can be transformed with

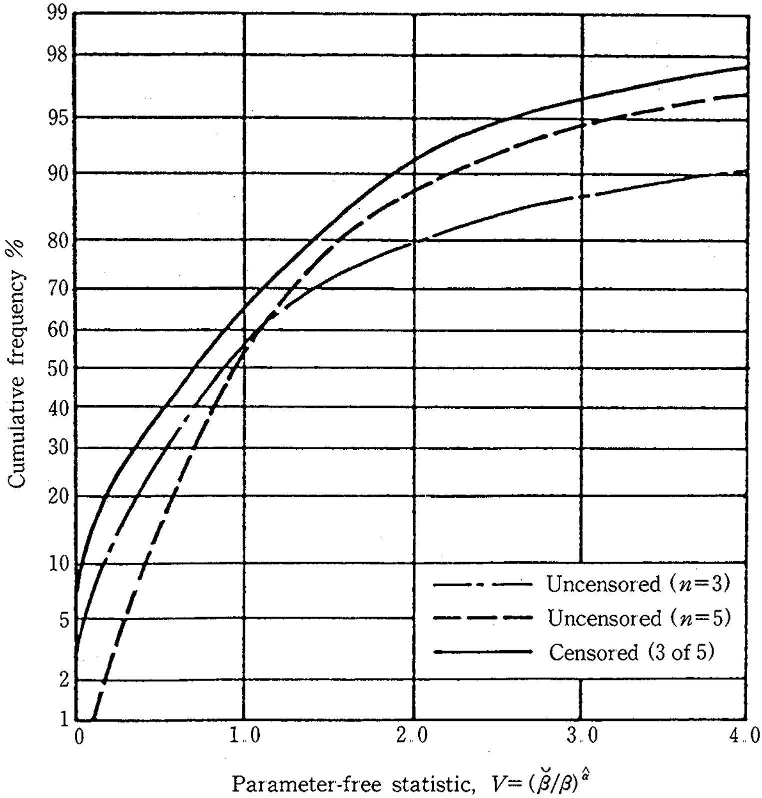


Fig. 8. 5 Comparison of the empirical distribution of the Weibull scale parameter β between complete and censored one

the aid of Eq. (7. 22) as follows:

$$W=(\hat{\beta}/\beta)^{\alpha}$$

$$=\left[\frac{1}{k}\left\{\sum_{i=1}^{k} t_i^{\alpha}+(n-k)t_k^{\alpha}\right\}\right]/\beta^{\alpha}$$

$$=\frac{1}{k}\left\{\sum_{i=1}^{k}\left(\frac{t_i}{\beta}\right)^{\alpha}+(n-k)\left(\frac{t_k}{\beta}\right)^{\alpha}\right\}$$

$$=\frac{1}{k}\left\{\sum_{i=1}^{k} y_i+(n-k)y_k\right\}$$

$$= \frac{1}{k}\left\{\sum_{i=1}^{k}(n-i+1)(y_i-y_{i-1})\right\} \tag{8.14}$$

When the complete outcomes $T_i = t_i$, $i = 1, 2, \ldots, n$ of size n of fatigue life T which follows Weibull distribution are given, the corresponding Y_i's of size n, $Y_i = y_i$, $i = 1, 2, \ldots, n$, follow the exponential distribution with mean $= 1$ respectively, and are parameter-free from α and β. For this reason, W given in Eq. (8.13) also becomes a parameter-free statistic, and besides the following variate

$$X_i = (n-i+1)(Y_i - Y_{i-1}) \quad (i = 1, 2, \ldots, n) \tag{8.15}$$

also follows the exponential distribution with mean $= 1$. Therefore, from Eq. (8.14), nW is given as the sum of the exponential variate of size n with mean $= 1$. By the way, the probability density of X_i is given as

$$f_{X_i}(x_i) = e^{-x_i} \tag{8.16}$$

On the other hand, the density function of the random variable $G_i (G_i = 2X_i)$ is

$$f_{G_i}(g_i) = f_{X_i}(x_i)\left|\frac{dg_i}{dx_i}\right|^{-1} = \frac{1}{2}e^{-g_i/2} \tag{8.17}$$

and the density function of χ^2 with ϕ degrees of freedom is

$$f(\chi^2) = \frac{1}{2^{\phi/2}\Gamma(\phi/2)}(\chi^2)^{\phi/2-1}\exp\left(-\frac{\chi^2}{2}\right) \tag{8.18}$$

Settig $\phi = 2$ in Eq. (8.18) reduces to Eq. (8.17), which means that G_i follows χ^2 distribution with 2 degrees of freedom. Therefore, from the reproducibility of χ^2 distribution,

$$2nW = \sum_{i=1}^{n} 2X_i = \sum_{i=1}^{n} G_i$$

will follow χ^2 distribution with $2n$ degrees of freedom, and hence the density function of W is given as follows:

$$f_W(w) = \frac{2n}{2^n\Gamma(n)}(2nw)^{n-1}\exp(-nw)$$

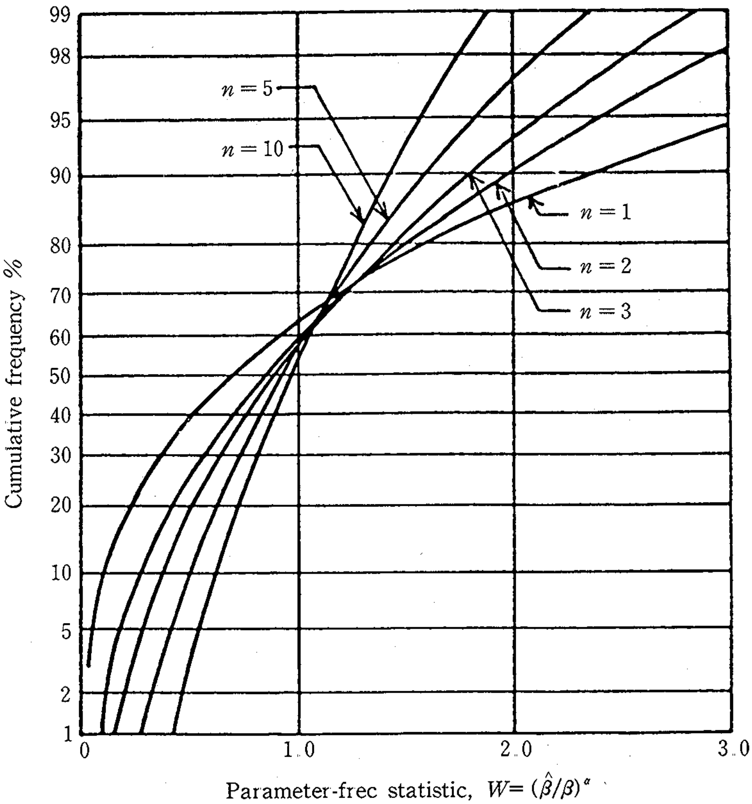


Fig. 8.6 Theoretical distribution of the MLE of the Weibull scale parameter β for complete samples when Weibull shape is assumed to be known.

$$= \frac{n^n}{\Gamma(n)} w^{n-1} \exp(-nw) \tag{8.19}$$

Fig. 8. 6 represents the theoretical distribution of the statistic W given by Eq. (8. 13) as a parameter of n. In this figure, the distribution function is shown in place of the density function. Compared with Fig. 8. 4 which represents the distribution of the MLE $\breve{\beta}$ when α is assumed to be unknown, the distribution of $\hat{\beta}$ shows less scatter around the true value α, and,

therefore, an estimate of β has considerably high reliability even when estimated based upon only one sample, that is, $n=1$. Similar to the distributions of U and V, an estimate $\hat{\beta}$ approaches the true value β with increasing sample size. It should be emphasized that when the Weibull shape α is known, the MLE $\hat{\beta}$ of the scale parameter β has less scatter in comparison to the corresponding MLE $\breve{\beta}$ for unknown shape.

9 Concluding Remarks

The Weibull probability model has been frequently used to express fatigue life distribution of a structural component. It plays an important role in the reliability-based design of machines and structures.

In the present paper, the parameter estimation of a two-parameter Weibull distribution has been discussed in detail, and the maximum likelihood estimators for the Weibull shape and scale parameters in case of both being unknown as well as that for Weibull scale in case of known shape are theoretically derived as the simultaneous solutions of a system of likelihood equations.

Unfortunately, however, the solutions cannot be obained in a closed form. In this respect, parameter-free statistics for the maximum likelihood estimators have been introduced and their empirical distributions have been established with the aid of Monte Carlo simulation techniques to clarify their statistical properties in connection with the true parameter values, the sample size and the censoring number. The results obtained in the present study are undoubtedly believed to be of crucial importance in the reliability-based design.

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