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A Proof of the Fundamental Theorem of Algebra via Reich's Coincidence Theorem and a Reason Why There Exists No Proof Based on a Simple Application of Brouwer Fixed Point Theorem

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Abstract.

This paper is to report that it is 'almost' impossible to prove the fundamental theorem of algebra (FTA) by a simple application of Brouwer fixed point theorem. While providing a new simple proof of the FTA based on Reich's coincidence theorem, we also show that it is necessary to prove beforehand the essentiality (the property of being not nullhomotopic) of the map z^n on the unit circle. Since Brouwer fixed point theorem is equivalent to the essentiality of the identity map, the FTA can be regarded as being topologically more sophisticated than Brouwer fixed point theorem for the two dimensional space. Besides, in our proofs of the FTA, we avoid the use of arguments based on homotopy and its related concepts, thus even high school students will have little difficulty in

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following the proofs, so long as the meaning of Brouwer fixed point is understood and accepted as the given starting proposition. We also mention a possible generalization of our result to the case of multivalued maps, and its application to Walrasian general equilibrium analysis.

Keywords : Brouwer fixed point theorem, essential map, fundamental theorem of algebra, homotopy, Schirmer coincidence theorem

JEL Classification : C 62, C 65

1 Introduction

There have been more and more proofs of the fundamental theorem of algebra (FTA) appearing in the literature, showing that many mathematicians are still eager to prove the FTA by use of a theorem or a technique depending on their own specialization in Mathematics, while the teachers have been wishing to devise a simpler and more elementary proof to be presented in classrooms. It is of basic importance that the content of the theorem are easily understood by most high school students. In simple terms the FTA states that every non-constant single variable polynomial with complex coefficients has a complex root.

Now, one might have expected that there have been, among more than 100 proofs, several simple proofs based upon a direct application of Brouwer fixed point theorem. (More properly, Bohl-Brouwer fixed point theorem because of the contribution by Bohl [4]. For a very simplistic argument, x is called a fixed point of a function $f(x)$ if $f(x)=x$ and if we take $g(x)$ as equivalent to $(f(x)-x)$, then x would be a solution for $g(x)=0$.) Surprisingly, however, we seem to have only one proof on these lines by Fort [8], an earlier one by Arnold [1] was unfortunately wrong. (See Arnold and Niven [2].) The use of Brouwer fixed point theorem in Fort [8], however, is neither simple nor direct. His paper is not mentioned in the classic reference - Fine and Rosenberger [7].

Mathematics teachers who wish to find a simple proof based on a fixed point theorem have been helped by Nakaoka [15] and Dodson [5], which used a coincidence theorem of Schirmer [17] or its variant. Needless to say her coincidence theorem is a generalization of Brouwer fixed point theorem. It would thus be all the more useful in the classroom if we should be able to present a proof by a simple application of Brouwer fixed point theorem. In the past, some of the authors of this paper have tried to get such, but have ended up with some wrong proofs, and now believe that it is impossible to prove the FTA by use of Brouwer fixed point theorem without employing a continuous n-th root of a given polynomial *a la* Fort [8], or more precisely without involving the essentiality of the map z^n on the unit circle. We have thus, had to change our strategy ; first to construct a generalization of Brouwer fixed point theorem, which is geometrically easy to understand, at least easier to remember and apply. We soon found this had already been done by Schirmer [17]. Moreover, the FTA had been proved by Dodson [5] based upon her theorem. On the other hand, Reich [16] generalized both Schirmer's theorem and fixed point theorems by Bohl [4], Shinbrot [18] and Kaniel [11]. We obtained our theorem by a simple application of Reich's coincidence theorem. In so doing, however, we have avoided the concepts of homotopy, fundamental group, winding number, and the index or degree of mapping, solely depending upon the essentiality of a map on a sphere defined as non extensibility of its domain to the entire ball. Moreover, we present our theorem, following Kulpa [12], in terms of a necessary and sufficient condition so that the role of essentiality comes forward inevitably.

This short paper is organized as follows. The next section explains some coincidence theorems, which are generalizations of Brouwer fixed point theorem, and then we present our theorem. Using our theorem, we give a new proof of the FTA in section 3. We discuss the essentiality of the identity map on the sphere as well as the map z^n in section 4. In section 5, a more difficult problem in which the coefficients of a polynomial can be a function of the variable is dealt with, and a proof of the existence of a solution is given under a condition which is more general than requiring constant coefficients. In the final section, some remarks are given, including why a simple application of Brouwer fixed point theorem cannot lead to a proof of the FTA, and how we can generalize our theorem to multivalued mappings, and use it in an existence proof of general equilibrium models à la Walras without Walras' law.

2 Coincidence Theorems and a New Theorem

First we explain our symbols. Let $\mathbb{R}^n$ be the real Euclidean space of dimension n, B_R^n the ball of radius $R > 0$ with its center at the origin, and S_R^{n-1} the boundary of B_R^n. In the following, we use the symbols B and S in place of B_R^n and S_R^{n-1} when there can be no misunderstanding.

We start with the definition of essential maps.

Definition. A continuous map f from S to itself is *essential* if and only if f has no continuous extension F from B to S.

Normally in textbooks on topology, the concept of null homotopy or inessentiality is first introduced, and the essentiality of a map is defined as being not nullhomotopic. The null homotopy of a map on S is then characterized by the existence of its continuous extension to B. (See, e. g., Dugundji [6, p. 316].) Our definition above is made by connecting the essentiality directly with non extensibility, thus dispensing with the concept of homotopy. We use the term *essential*, simply because it is short and was used in the past. From the above definition itself, we have

Theorem 2. 1. There exists no continuous map f from B to S such that $f|_S$ is essential.

This theorem is a generalization of the no retraction theorem where f is the identity map. The essentiality of the identity map on the sphere is discussed below in section 4. Using this theorem, we can establish Schirmer's coincidence theorem.

Theorem 2. 2 (Schirmer [17]). Let f and g map B continuously into itself, and suppose that $f(S) \subset S$. If $f|_S : S \to S$ is essential, then f and g have a coincidence.

Proof. If there exists no coincidence point, we can construct a map from B to S, by transforming a point x in B to a point on S where the ray from $g(x)$ to $f(x)$ crosses with S. This map is continuous and identical with f on S, which is essential, contradicting Theorem 2. 1. □

This proof has already been suggested by Dodson [5, p. 478]. It is clear that Brouwer fixed point theorem is a special case of Schirmer's theorem above, when f is the identity map. Since any constant map, say $g(x) = x^\circ$ for any $x \in B$, mapping to the same point $x^\circ \in B$, is continuous, the map f here is onto B because f has a coincidence with g by Theorem 2. 2. Note also that two maps f and g both have its own fixed points in B.

Now we can also prove Reich's more general coincidence theorem based on Schirmer's, thus avoiding the use of homotopy: Reich [16] employed a homotopy in his simple and elegant demonstration.

Theorem 2. 3 (Reich [16]). Let f and g map B continuously into $\mathbb{R}^n$, and suppose that $f(S) \subset S$. If $f|_S : S \to S$ is essential and $g(x) \neq m \cdot f(x)$ for all $x \in S$ and $m > 1$, then f and g have a coincidence in B.

Proof. First we construct a map F as $F(x) \equiv \|x\| \cdot f(x) / \max(R, \|f(x)\|)$. This map is surely continuous, and $f(S) = f(S) \subset S$, with $F|_S : S \to S$ being essential. Next, we suppose there is no coincidence between f and g in B, and define a second map $G \equiv R \cdot (g(x) - f(x)) / \|g(x) - f(x)\|$. This map G is also a continuous map from B to S. Thus, we can apply Schirmer's theorem, and there should exists a coincidence $x^* \in B$ such that $F(x^*) = G(x^*)$. We should have $x^* \in S$ because $G(x^*) \in S$ and $F(x^*)$ can be on S only when x^* is on S. Now the equality $F(x^*) = G(x^*)$ implies that

$$f(x^*) = \alpha \cdot (g(x^*) - f(x^*)),$$

with α being a positive real. Thus, we have $g(x^*) = (1 + \frac{1}{\alpha}) \cdot f(x^*) = m \cdot f(x^*)$ with $m > 1$, and $x^* \in S$, yielding a contradiction. □

Theorem 2. 4. Let f and g map B continuously into $\mathbb{R}^n$, and suppose that $f(S) \subset S$. If $f|_S : S \to S$ is essential and $g(x) \neq m \cdot f(x)$ for all $x \in S$ and $m < 1$, then f and g have a coincidence in B.

Proof. In the proof of Theorem 2. 3, we change $G \equiv -R \cdot (g(x) - f(x)) / \|g(x) - f(x)\|$. In this case, the equality $F(x^*) = G(x^*)$ implies that

$$f(x^*) = -\alpha \cdot (g(x^*) - f(x^*)),$$

with α being a positive real. Thus, we have $g(x^*) = (1 - \frac{1}{\alpha}) \cdot f(x^*) = m \cdot f(x^*)$ with $m < 1$, and $x^* \in S$, a contradiction. □

Theorem 2. 3 can be regarded as a generalization of a fixed point theorem by Halpern and Bergman [10] for the case of inward maps, while Theorem 2. 4 for the case of outward maps, though the domain of maps here is not a compact convex set, as in general, but a sphere. Note again that the map f in Theorems 2. 3 and 2. 4, though allowed to bring interior points of B to its outside, is after all onto B, which is obvious from Theorem 2. 3, considering a coincidence with a constant map g into B. Note also that the map g in Theorems 2. 3 and 2. 4 either has a fixed point in the interior of B or has a coincidence with f on S or both. This implies in particular that if $g(S) \equiv \{y \mid y = g(x) \text{ for } x \in S\} \cap S = \emptyset$, then g has a fixed point in the interior of S.

Now we state and prove a new theorem, which is an extension of a result in Shinbrot [18, Corollary 2, p. 257] and Kaniel [11, Theorem 1, p. 259].

Theorem 2. 5. Let f and g map B continuously into $\mathbb{R}^n$, and suppose that $f(S) \subset S$. Then, for g which satisfies the condition that $g(x) \neq m \cdot f(x)$ for all $x \in S$ and $m < 0$ (or $m > 0$), the equation $g(x) = 0$ has a solution in B if and only if $f|_S: S \to S$ is essential.

Proof. First suppose that $f|_S: S \to S$ is essential, and consider the map $G(x) \equiv f(x)+g(x)$. This G satisfies the conditions given in Theorem 2.4 (or Theorem 2.3), thus f and G have a coincidence point x^* in B, implying $f(x^*) = f(x^*) + g(x^*)$. That is, $g(x^*) = 0$. Next suppose that $f|_S: S \to S$ is not essential. We can find a continuous extension of f, say F, from B to S. Let $g \equiv F$ (or $-F$), this g satisfies the specified condition, and yet the equation $g(x) = 0$ has no solution in B. □

Note that the map f itself satisfies the conditions on g, and so there exists an $x^\circ \in B$ such that $f(x^\circ) = 0$. Moreover, we notice that the map $g(x) \equiv f(x) - \overline{x}$, where $\overline{x}$ is an arbitrary point in the interior of B, satisfies the conditions in Theorem 2.5, thus there is an $x^\circ \in B$ such that $g(x^\circ) \equiv f(x^\circ) - \overline{x} = 0$, implying f is onto B by its continuity.

Theorems from 2.2 to 2.4 can also be restated in the style of 'if and only if' as in Theorem 2.5. In this note, however, they are presented as originally shown by the respective authors.

To memorize Theorem 2.5 visually, the reader may consider the geometrical meaning of the condition that $g(x) \neq m \cdot f(x)$ for all $x \in S$ and $m < 0$. This says simply that for any x on S, $g(x)$ is never on the reverse direction of $f(x)$. For example, when $g(x)$ for $x \in S$ is always in the half space $\{y \mid (y, f(x)) \geq 0, y \in \mathbb{R}^n\}$, where $(,)$ stands for the inner product, the condition is well satisfied.

3 A New Proof of the Fundamental Theorem of Algebra

Let $\mathbb{C}$ be the field of complex numbers, and we consider a monic polynomial $p(z) \equiv z^n + a_1 z^{n-1} + \cdots + a_n$, with the coefficients $a_i \in \mathbb{C}$, $n \geq 1$, and the variable $z \in \mathbb{C}$. The symbol $\mathbb{C} \setminus 0$ stands for the set of all nonzero complex numbers. The variable z is also written as $|z| \cdot e^{i\theta}$, with $0 \leq \theta < 2\pi$. We define $f(z) \equiv |z| \cdot e^{in\theta}$,

i. e., $f(z)$ is a transformation of z in the direction (argument) of z^n, but keeping the original absolute value. For a real $R > 0$, we define the disk $D_R \equiv \{z \mid |z| \leq R\}$ and the circumference $C_R \equiv \{z \mid |z| = R\}$.

The theorem we are going to prove is :

Theorem 3. 1. Every complex, non-constant polynomial has a zero.

Proof. It is evident that we can concentrate on the case of non-constant monic polynomials. We suppose that $p(z) \neq 0$ for $z \in \mathbb{C}$, implying $a_n \neq 0$. It is also clear that we can regard D_R as B_R^2, and C_R as S_R^1, or simply S. When $|z|$ is large enough, say $\overline{R}(> 1)$, the directions of $p(z)$ and $f(z)$ are more or less in the same direction because $\lim_{|z| \to \infty} p(z)/z^n = 1$. Thus, we can choose $\overline{R}$ large enough so that $p(z) \neq m \cdot f(z)$ for all $z \in C_{\overline{R}}$ and $m < 0$. It is proved in the following section that $f|_{C_{\overline{R}}}$ is essential for any $R > 0$. We can now apply Theorem 2. 5, obtaining the existence of a solution to $p(z) = 0$. □

We can make explicit the size of $\overline{R}$. Indeed, if $\overline{R} \equiv \max\{1, \sum_{i=1}^{n} |a_i|\}$, we have

$$|\sum_{i=1}^{n} a_i \cdot z^{n-i}| \leq \sum_{i=1}^{n} (|a_i| \cdot |z|^{n-i}) \leq (\sum_{i=1}^{n} |a_i|) \cdot |z|^{n-1} < |z|^n \text{ for } z \in C_{\overline{R}},$$

which is given, e. g., in Fine and Rosenberger [7, p. 200]. Since $f(z) \equiv |z| \cdot e^{in\theta} = \alpha(z) \cdot z^n$ with $\alpha(z) > 0$ for $z \neq 0$, the above inequality guarantees that $p(z) \neq m \cdot f(z)$ for all $z \in C_{\overline{R}}$ and $m < 0$. Thus we have a better bound for solutions than in Arnold [1] and in Fort [8], albeit a well-known one (Goursat [9, p. 104]).

Given $\overline{R}$ above, we can prove more that the inner product between $p(z)$ and $f(z)$ is positive for $z \in C_R$, that is, the angle between them is less than $\pi/2$. (Here, the inner product stands for the one when the complex plane is regarded as the two dimensional Euclidean space.) This is because

$$\mathrm{Re}(p(z))\cdot\mathrm{Re}(f(z))+\mathrm{Im}(p(z))\cdot\mathrm{Im}(f(z)) \geq |z|^{n+1}-|\sum_{i=1}^{n} a_i\cdot z^{n-i}|\cdot|z|$$
$$= |z|\cdot(|z|^n-|\sum_{i=1}^{n} a_i\cdot z^{n-i}|)>0.$$

In other words, $p(z)$ for $z\in C_{\overline{R}}$ is always in the half space $\{y\mid(y,\ f(x))\geq 0,\ y\in\mathbb{C}\}$.

4 Essential Maps

To apply our Theorems 2.2 to 2.5, we need to know certain maps are essential. Given Brouwer fixed point theorem, or its equivalent, i.e., the no retraction theorem, it is easy to prove that the identity map on S_R^{n-1} is essential. The negation of essentiality of the identity map implies the existence of a continuous extension map from B_R^n to S_R^{n-1}, which contradicts the no retraction theorem. If the reader wishes to use Brouwer fixed point theorem, the antipodal map can in addition be applied, as is well known. Thus, the essentiality of the identity map on S_R^{n-1} is equivalent to Brouwer fixed point theorem.

Now it seems more difficult to prove that the map z^n is essential on C_1. And at the same time in this note we want to shrink from the use of degree of mappings and its related concepts. Fort [8] has, however, already presented an elementary proof of a theorem which can be employed to show the essentiality of z^n on C_1. Fort's Theorem 5 in [8, p. 374] asserts that if f is a continuous map from a disk D_R into $\mathbb{C}\setminus 0$ and n is a positive integer, then there exist n distinct continuous maps $h_1,\ \dots,\ h_n$ which have the property that $[h_k(z)]^n=f(z)$ for all $z\in D_R$. That is, continuous n-th roots can be found if a given continuous function on D_R does not include the origin in its image. Suppose that the map z^n is not essential on C_1. Then there is a continuous extension F from D_1 to C_1 with $F|_{C_1}=z^n$. Surely F does not have the origin in its image of D_1, thus we can pick up one of the n-th root function, say $h_0(z)$ such that $h_0(1)=1$. Now the map $h_0\cdot F$ is a continuous function from D_1 to C_1, and on the C_1, it is the identity, showing a contradiction to the no retraction theorem. Therefore the map z^n is essential on C_1. Since C_1 and C_R are homeomorphic for any $R>0$, the map $f|_{C_{\overline{R}}}$ in the previous section is also essential on C_R. (Concerning continuous logarithm, the reader is referred to Ash and Novinger [3].)

5 A Slightly More Difficult Problem

Our proof of the FTA does not make use of the fact that polynomials are holomorphic, while most analytic proofs need this property. Thus our method can also work in more general cases where a given function is not holomorphic but simply continuous.

Let us consider a polynomial-type function in which the coefficients can be a function of the variable, or more precisely, a given function is expressed as

$$p(z)\equiv z^n+a_1(z)\cdot z^{n-1}+\cdots+a_{n-1}(z)\cdot z+a_n(z) \text{ for } z\in\mathbb{C}.$$

Here the coefficients $a_i(z)$'s are continuous functions of z, and not necessarily holomorphic in D_R. Now we have

Theorem 5. 1. A polynomial-type function in which the coefficients can be a continuous function of the variable, as described above, has a zero, provided $|a_i(z)|\leq\max(K_i,\ A_i\cdot|z|^{i-1})$ for $z\in\mathbb{C}$ with reals $K_i>0$, $A_i>0$ for $1\leq i\leq n$.

Proof. Almost the same proof as in Theorem 3. 1 can be applied. We choose $\overline{R}\equiv\max\{1,\ \sum_{i=1}^{n}\max(K_i,\ A_i)\}$, and we have

$$|\sum_{i=1}^{n}a_i(z)\cdot z^{n-i}|\leq\sum_{i=1}^{n}(|a_i(z)|\cdot|z|^{n-i})\leq(\sum_{i=1}^{n}\max(K_i,\ A_i))\cdot|z|^{n-1}<|z|^n$$

$$\text{for } z\in C_{\overline{R}},$$

which is enough to show that $p(z) \neq m \cdot f(z)$ for all $z \in C_{\overline{R}}$ and $m < 0$. □

It seems that any of the existing complex analytical proofs cannot easily be adapted to prove Theorem 5.1. In more general, when we have $\lim_{|z|\to\infty} p(z)/z^n = c \neq 0$ for $c \in \mathbb{C}$, $p(z) = 0$ has a solution so long as $p(z)$ is continuous on $\mathbb{C}$. This is clear from our Theorem 2.5, because $p(z)$ and $f(z)$ have their arguments with an almost fixed difference, i.e., that of c, when $|z|$ becomes large, implying thus the impossibility of either $p(z) \neq m \cdot f(x)$ for all $z \in C_{\overline{R}}$ and $m < 0$, or $m > 0$.

6 Concluding Remarks

Remark 1. With hindsight, we can simplify the proof in Fort [8], sharpening the bound of solutions at the same time. Our starting point is again Fort's Theorem 5 ([8, p. 374]. As in the above, let $R \equiv \max\{1, \sum_{i=1}^{n} |a_i|\}$. Then, $f(z) \equiv |z| \cdot e^{in\theta}$ and $p(z)$ for $z \in D_R$ have their arguments which differ less than $\pi/2$. Now if $p(z) \neq 0$ for $z \in D_R$, there is continuous n-th roots of $p(z)$, i.e., $\sqrt[n]{p(z)}$, as is explained in the preceding section. It follows that z and $\sqrt[n]{p(z)}$ have the arguments difference still less than $\pi/2$ for $z \in D_R$. Then we can conceive a continuous map g from D_R to C_R by defining $g(z)$ as the point on C_R where the ray from z to $z + \sqrt[n]{p(z)}$ intersects with C_R. On C_R, the map g is the identity, and so we have a contradiction to the no retraction theorem. Thus there should be a $z^* \in D_R$ such that $p(z^*) = 0$.

Remark 2. Kulpa [12] presented an elementary proof for Schirmer's coincidence theorem based on a special case of Stokes theorem, and discussed its application to the FTA following Dodson [5]. He also showed that the converse of Schirmer's theorem is also valid. That is, let f be a continuous map from S to S. If for arbitrary two continuous maps F and G from B to B such that $F|_S = f$, there exists a point $x \in B$ such that $F(x) = G(x)$, then f is essential. It is easy to prove this by contraposition. If f is not essential, there is a continuous extension F of f, which maps B to S. We adopt the constant map from B to its center as G. Evidently there is no coincidence between these two maps, F and G. This proposition tells us that the essentiality of a map in general is as highly sophisticated as Schirmer's coincidence theorem. It is noted above that Brouwer fixed point theorem is equivalent to the essentiality of the identity map on S, while the FTA is related to the essentiality of the map z^n on C_1. Thus if we stick to topological proofs of the FTA by use of fixed point or coincidence theorems, depending on only the continuity of a given map, i. e., making no use of the fact that a polynomial function is holomorphic, it is impossible to obtain a proof by a simple and direct application of Brouwer fixed point theorem: such a proof needs an additional contrivance equivalent to a demonstration of the essentiality of the map z^n. Our Theorem 2. 5 has been formulated so that this impossibility can been seen clearly.

Remark 3. In Walrasian general equilibrium analysis, it is desirable to weaken the requirements of Walras' law as well as the homogeneity concerning the excess demand functions. The reader is referred to Lehmann-Waffenschmidt [13] and Maskin and Roberts [14]. Our method shall be based on an extension of Theorem 2. 5 to the case of multivalued maps. In a sense, the result due to Shinbrot [18, Corollary 2, p. 257] can deal with general equilibrium models withWalras' law but without the homogeneity, while its generalization by Kaniel [11, Theorem 1, p. 259] can dispense with both the law and the homogeneity. Some generalizations of this sort will constitute the authors' next joint paper, and will indeed provide still another proof of the FTA.

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