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A Generalization of Theorems on Inverse-Positive Matrices*

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Abstract

A generalization is presented concerning theorems on non-negative invertibility. That is, a necessary and sufficient condition is given under which a real square matrix has its inverse with non-negative columns in a subset of the whole index set. This condition is simple, geometrical, and intuitively natural. The result includes as a special case various theorems on non-negative invertibility in the ordinary sense. Motivation of the generalization from economics as well as its significance are also explained.

Keywords: Non-negative invertibility; Columnwise partial non-negativity; Geometrical conditions

1 Introduction

In Dasgupta and Sinha(1979), a sufficient condition is presented for a real square matrix to have the non-negative inverse, which condition is easily modified also to be necessary. The publication of this paper was delayed until Dasgupta and Sinha(1992) and Dasgupta(1992) appeared. Bidard(1991, p.113; 1996), Erreygers(1996), and Bidard and Erreygers(1998) rediscovered the proposition due to Dasgupta and Sinha. (See Berman and Plemmons's book(1979) for non-negative invertibility of M-matrices.) In this note, we present a necessary and sufficient condition under which the columns of the inverse in a certain subset of the index set are non-negative, thus generalizing the result by Dasgupta and Sinha. The method of proof is simple and elementary.

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In section 2, we present our condition and the theorem together with two corollaries, then section 3 explains how this generalization comes from linear economic models with durable machines. In section 4 some numerical examples are raised. Finally in section 5, we conclude with some remarks.

2 Propositions

First we explain our notation and assumptions. The symbol R^n means the real Euclidean space of dimension $n (n \geq 2)$, and R^n_+ the non-negative orthant of R^n. A given real $n \times n$ matrix M maps from R^n into itself. Let $(M)_{ij}$ represent the (i, j) entry of M, while $x \in R^n$ stands for a column vector, x^t for the transpose of x, and x_i for the i-th element of x. The symbol $(M)_{.j}$ means the j-th column of M. One more symbol $\mathbf{1}$ is the column vector R^n whose entries are all unity. The inequality signs for vector comparison are as follows:

$x \geq y$ iff $x - y \in R^n_+$;
$x > y$ iff $x - y \in R^n_+ - \{0\}$;
$x \gg y$ iff $x - y \in \text{int}(R^n_+)$,

where $\text{int}(R^n_+)$ means the interior of R^n_+. These inequality signs are used also for matrices in a similar meaning.

The index set S is a non-empty subset of the whole index set $N \equiv \{1, 2, 3, ..., n\}$. One more index set T is the complement of S with respect to N. We allow T to be empty. Now we state two assumptions.

A1. There exists an $x \in R^n_+$ such that $(Mx)_j > 0$ for $j \in S$, and $(Mx)_j = 0$ for $j \in T$.

A2. If, for $x \in R^n_+$, we have $(Mx)_j > 0$ for $j \in S$, and $(Mx)_j = 0$ for $j \in T$, then $x \gg 0$.

The assumptions A1 and A2 have clear geometrical as well as economic interpretations. When the hyper-plane $H \equiv \{x \mid x \in R^n,\ x_i = 0 \text{ for } i \in T\}$, with $H_+ \equiv H \cap R^n_+$, and $\text{r\_int}(H_+) \equiv \{x \mid x_j > 0 \text{ for } j \in S, \text{ and } x_i = 0 \text{ for } i \in T\}$ is considered, the assumption A1 requires that the image of R^n_+ by M intersects with $\text{r\_int}(H_+)$, while A2 demands that $\text{r\_int}(H_+)$ should be covered by the image of $\text{int}(R^n_+)$ by M. Concerning economic interpretations of the assumptions, a more detailed account is given in the next section.

In the sequel, a vector whose existence is guaranteed in the condition A1 is denoted by x^*, and we must have $x^* \gg 0$ by A2. Now we can prove

Theorem 2.1. A given matrix M has the following three properties if and only if the two assumptions A1 and A2 are satisfied for M.

(i) M is non-singular;

(ii) the j-th column of the inverse M^{-1} is non-zero and non-negative for every $j \in S$.

(iii) for each $i \in N$, there exists at least one $j \in S$ such that $(M^{-1})_{ij} > 0$.

Proof. The "only if" part is evident, and so we show the "if" part only. First let us prove that M is non-singular. Suppose to the contrary that $Mx = 0$ for

$x \neq 0$. We consider two cases. The first case is when $x > 0$. Construct a vector $y \equiv x^* - kx$, where $k \equiv \min_{x_i \neq 0}(x_i^*/x_i)$. This vector y is non-zero non-negative and has at least one zero element, thus gives a contradiction to A2. The second case is when x includes one or more negative elements. In this case we construct a vector $y \equiv x^* + kx$, where $k \equiv \min_{x_i < 0}(x_i^*/(-x_i))$. Again in the second case, this y is non-zero non-negative and contains at least one zero element, contradicting A2. Therefore, M must be non-singular. Now let us suppose, without losing generality, that the set S contains the index 1, and M^{-1} contains a negative element in the first column. Then, consider the solution to $Mu = d$, where d is a vector such that $d_j > 0$ for $j \in S$, and $d_j = 0$ for $j \in T$. When d_1 is large enough, the solution gives a contradiction to the assumption A2. Certainly, the property (iii) should be satisfied by the assumption A2. □

We now present two corollaries.

Corollary 2.2. When M is a matrix of the form $\lambda I - A$, where A is a real $n \times n$ non-negative matrix and λ is a positive number greater than the Frobenius root of A, it is inverse-positive.

Proof. It is not difficult to see both the assumptions A1 and A2 are satisfied with the set T being empty. Thus, all the columns of its inverse is non-zero and non-negative.□

Corollary 2.3. When a given matrix M satisfies the assumptions A1 and A2 with a certain non-empty set T, and it cannot satisfy A1 and A2 for any index set which is a proper subset of T, including the empty set, then $(M^{-1})_{\cdot j}$ for $j \in T$ has at least one negative element.

Proof. It is obvious from our theorem above and the very assumptions of A1 and A2.□

3 Motivation : Input-Output Models with Fixed Capital

Let us begin with a Leontief model of circulating capital without joint production:

$$x = (1 + g)Ax + d, \tag{3.1}$$

where $x \in R_+^n$ is a column vector of activity levels, $d \in R_+^n$ a column vector of final demand, A an $n \times n$ material input coefficient matrix, and g a rate of steady balanced growth. Each column of A represents the inputs of various commodities to a production process operated at a unit activity level, while each row stands for a vector of inputs of a particular commodity to the whole set of processes. One more symbol $\mathbf{1}$ is the column vector of labor input coefficients, and it is assumed that every process requires labor and is normalized so as to make each labor input coefficient unity, i.e., $\mathbf{1}^t \equiv (1, 1, \ldots, 1)$. When A and g

are given, (3.1) can be solved for a non-negative x for any $d \in R^n_+$ if and only if $(1+g) < \frac{1}{\lambda(A)}$, with $\lambda(A)$ standing for the Frobenius root of A. (We assume $\lambda(A) > 0$.) In sum, $(I-(1+g)A)^{-1} > 0$. And $p \equiv \mathbf{1}^t \cdot (I-(1+g)A)^{-1}$ can be interpreted as the equilibrium price vector with g now meaning the uniform rate of profit.

When we allow for joint production, the relevant equation is then:

$$Bx = (1+g)Ax + d, \tag{3.2}$$

where B is an $n \times n$ material output coefficient matrix. Suppose again that B, A, and g are given as data, then (3.2) has a solution $x \in R^n_+$ for any $d \in R^n_+$, if and only if the matrix M defined as $M \equiv B - (1+g)A$ satisfies our assumptions A1 and A2 with $T = \emptyset$. In general, however, there is no good reason why a model with joint production satisfies the assumptions A1 and A2, and so (3.2) may not have a non-negative solution for final demand vectors in some subset of R^n_+. The assumption A1 may be named the "productivity" condition, and A2 the "essentiality" condition.

Here some words are in order to explain why the matrices A and B can be square. When the number of processes exceeds that of commodities, we differentiate a commodity produced by two or more different processes, thus increasing the number of commodities. This operation may cause an apparent problem in the calculation of equilibrium prices. Competition, however, eliminates those processes one by one which make losses when the price minimum among all differentiated varieties of a commodity is adopted. (There can be another problem: square matrices thus dynamically obtained may not be unique.) See Schefold(1978b) for explanation on how the squareness appears after optimization, not before production decisions.)

Now, in between two systems (3.1) and (3.2), we have models of fixed capital(called *machines* below) without *proper* joint production. When old machines can be converted to a certain amount of the new machine in terms of efficiency unit, i.e., the case of *quantitative* depreciation, the model is essentially that of single production. If old machines are qualitatively different from new ones, they should be thought of as joint products. Though old machines can thus be regarded as a kind of joint products, (3.2) may have a non-negative solution because they appear in a special manner: to have an old machine in output, a machine younger by one year must be among inputs. Moreover, old machines need not turn up in the final demand vector unless they can be exported abroad. Let us consider a concrete example of an economy, where there are two commodities: the first one is a consumption good, and the second a machine. A machine can be used physically for two years, and we regard a one year old machine the third commodity. The material coefficient matrices are described as

$$B \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \mathrm{A} \equiv \begin{pmatrix} 0 & 0.1 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The first process uses one new machine to produce one unit of consumption good plus one year old machine, the second uses one old machine and 0.1 unit of consumption good, for its maintenance, to produce one unit of consumption good, and the third produces one new machine by use of 0.5 unit of consumption good. The net output matrix $(B-A)$, when the rate of growth is zero, is given as $M3$ in the next section. As can be seen in the next section, the inverse of $M3$ has non-negative vectors in the first two columns. It is easy to see the assumptions A1 and A2 are satisfied for $M3$ with $T \equiv \{3\}$. This is a proposition treated in Kurz and Salvadori(1995, Chaps.7 - 9), and they therein presented a set of conditions sufficient to guarantee our A1 and A2, most notably the absence of proper joint production and non-shiftability of old machines among the processes. It should be noted that our assumptions A1 and A2 do not exclude joint production completely.

The problem is how to keep the relevant matrices square. Assume that old machines of a particular type are not transferable to other processes, and the old machine of a certain age add to the number of commodities. Then, we can perceive one more process which uses this old machine exclusively in place of its brand new one to perform the same production. This method of increasing processes *pari passu* with commodities is not affected by the existence of plural types of machines. When, however, old machines are transferable to other industries, they are truly proper joint products, and it may bring forth a problem, not to the squareness of matrices, but to the assumption A2. The squareness can be kept by regarding old machines installed in different industries as distinct, and this is not illegitimate because the installation of an old machine in an industry different form the one where it was employed needs normally some inputs of commodities. The assumption A2 may be violated since an old machine can have many places to be consumed, making some processes *not essential.*

We shift our topic to the equilibrium prices. They can be obtained by $p = \mathbf{1}^t \cdot (B-(1+g)A)^{-1}$, with g meaning the rate of profit. The example $M3$ has $p = (2,2,1)$, when $g = 0$. The inverse, however, includes negative elements in the columns belonging to the set T. Therefore, p_j, $j \in T$, that is, the price of an old machine can be negative. When this takes place, inefficiency in the use of old machines lies implicit. This problem should be treated in more general models consisting of inequalities rather than equalities, and yet the system of equalities may detect the existence of inefficiency. For example, when the $(1,2)$ entry of $M3$ becomes 0.25, i.e., less productive, the price vector becomes $p = (4,3,0)$, and the entry gets smaller than 0.25, still less productive, the price of the old machine becomes negative, and the second process involving the old machine should not be used.

4 Numerical Examples

When M is an M-matrix, which case is summarized as Corollary 2.2 in the above, it is inverse-positive as is well known. Thus no numerical example is necessary.

Now when $M1$ is given as

$$M1 \equiv \begin{pmatrix} 1 & -a & 1 \\ 1 & 1 & -a \\ -a & 1 & 1 \end{pmatrix},$$

then it has the inverse

$$\begin{pmatrix} \frac{-1}{a^2-a-2} & \frac{-1}{a^2-a-2} & \frac{-a+1}{a^2-a-2} \\ \frac{-a+1}{a^2-a-2} & \frac{-1}{a^2-a-2} & \frac{-1}{a^2-a-2} \\ \frac{-1}{a^2-a-2} & \frac{-a+1}{a^2-a-2} & \frac{-1}{a^2-a-2} \end{pmatrix},$$

when $a \neq -1$ or $a \neq 2$. This inverse is non-zero and non-negative if $1 \leq a < 2$. It is easy to generalize the above example $M1$ to a real square matrix with an odd number of columns and rows. When a real matrix has an odd number ($2k+1$ with $k \geq 1$) of columns and rows, its diagonal elements are all unity, and in each row, shifting from the diagonal element to the right, $(-a)$ and 1 appear alternately(jumping back to the first element at the rightmost one), then it is inverse-positive when $1 \leq a < (1+\frac{1}{k})$. (Each element of the inverse contains a fraction whose denominator is $(ka^2 - a - (k+1))$, while its numerator is either $((k-1)a - k)$ or $(-a+1)$.) It is not hard to see the assumptions A1 and A2 are satisfied with $T = \emptyset$. Therefore, for example

$$M2 \equiv \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

is inverse-positive.

Next we raise an example discussed in the previous section, for which the set T is not empty. That is, the following $M3$, with $T \equiv \{3\}$.

$$M3 \equiv \begin{pmatrix} 1 & 0.9 & -0.5 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

This has the inverse: $\begin{pmatrix} 0.7142\ldots & 0.3571\ldots & 0.6428\ldots \\ 0.7142\ldots & 0.3571\ldots & -0.3571\ldots \\ 0.7142\ldots & 1.3571\ldots & 0.6428\ldots \end{pmatrix}$.

As is explained in Section 3, it is easy to verify the assumptions A1 and A2 for $M3$.

5 Concluding Remarks

It has been made clear in Shefold(1978a) and Herrero and Villar(1988) that the so called non-substitution theorem can survive even in models of joint production so long as an optimal pair of coefficient matrices B and A can produce the non-negative inverse $(B-A)^{-1}$. In fact, the matrices need not be square, and a greater number of commodities than processes is allowed for. In fact, our theorem here is used to present a complete characterization of economies which enjoy the nonsubstitution property in Fujimoto, Herrero, Ranade, Silva, and Villar(2002), using the duality theorem in linear programming, i.e., the method of proof by Chander(1974).

One conjecture to be explored is that whenever negative prices appear, inefficiency is involved somewhere, and a machine of some age which has a negative price for the first time in its life should be discarded even when older ones may have positive prices. Although efficiency had better be handled in systems of inequalities rather than those of equalities, inefficiency may be detected within the latter.

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