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研究ノート

Schur Complements in Banach Spaces

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Abstract. In this research note, we extend the Duncan identity involving Schur complement to Banach spaces, and derive some propositions for the case where a given linear transformation is an M-operator. These results are natural generalizations of Le Chatelier-Samuelson principle discussed by Morishima, to Leontief models with an infinite number of commodities and processes.

Keywords: Banach spaces, Duncan identity, Le Chatelier-Braun-Samuelson principle, M-operators, Schur complements

1 Introduction

In Fujimoto and Ranade[10], two characterizations are presented concerning inverse-positive matrices. One is related to the well-known Hawkins-Simon condition as modified by Nikaido and Georgescu-Roegen. The other is based on a consideration of the Le Chatelier-Braun principle in thermodynamics, and it is asserted that each element of the inverse of the $(n-1) \times (n-1)$ principal submatrix is smaller than or equal to the corresponding element of the inverse of the original $n \times n$ matrix. When this result was extended to principal submatrices of smaller ranks in Fujimoto, Hisamatsu and Ranade[11], it was found that the key identity was already established long ago by Duncan[6], and it was closely related to the Schur complement so named by Emilie V. Haynsworth, and to the Banachiewicz identity. (For Schur complements, see a survey article by Ouellette[14]. For the original paper and later development, see Schur[16], Banachiewicz[1], Brezinski[2], Brezinski, Morandi Cecchi, and Redivo-Zaglia[3], Carlson[4], Cottle[5], Frazer, Duncan, and Collar[7], and Galántai[12]. For recent applications of Schur complements in numerical solution of differential equations, just make search in the WWW: there are many contributions.)

In economics, Morishima[13] analyzed Leontief models in a detailed way to obtain the propositions in comparative statics/dynamics, which Morishima called the Le Chatelier-Samuelson principle since they are connected with the Le Chatelier-Braun principle in thermodynamics and first introduced to economics by Samuelson employing the negative definiteness of Hessians or stability conditions.

This note is to extend a proposition in Fujimoto, Hisamatsu and Ranade[11] to Banach spaces, and establish a couple of theorems when a given linear map is an M-operator by combining two methods: the first one in Fujimoto, Hisamatsu and Ranade[11] decomposes the operator based on domains, and the other in Fujimoto, Herrero and Villar[9] employs an iterative method to show the existence of a solution.

Section 2 explains our basic notation, then in Section 3 our main theorems are presented. Section 4 gives concluding remarks.

2 Notation

Let D be a measure space, and X a Banach space of measurable functions on D. Any function space can be dealt with so far as it satisfies our assumptions below, e.g., $C(D)$, $L_2(D)$, and $L_\infty(D)$ can be handled. The set D is divided in two measurable spaces S and T such that $S \cup T = D$, $S \cap T = \emptyset$, $S \neq \emptyset$, and $T \neq \emptyset$. The space X has its nonempty closed convex pointed cone X_+ consisting of nonnegative functions on D, and a partial order $\leq$ (or $\geq$) introduced by X_+. A given linear operator M maps the space X into itself. The identity maps on X, on those functions restricted to S, and those restricted to T are denoted respectively by I_X, I_S, and I_T. The symbol N means the set of all natural numbers, $N \equiv \{1, 2, \dots\}$.

We consider a decomposition of linear map M as

$$M \equiv \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

by restricting the domain and the image set to S or T. When M is invertible, it is also decomposed as

$$M^{-1} \equiv \begin{bmatrix} \widehat{E} & \widehat{F} \\ \widehat{G} & \widehat{H} \end{bmatrix}.$$

3 Propositions

In this section we first state a general theorem concerning the Duncan identity.

Theorem 1. If M, E and $\widehat{H}$ are invertible, we have $\widehat{E} - E^{-1} = \widehat{F}(\widehat{H})^{-1}\widehat{G}$ or $E^{-1} = (A^{-1}/\widehat{H}) \equiv \widehat{E} - \widehat{F}(\widehat{H})^{-1}\widehat{G}$.

Proof. We adapt the proof for Theorem 3.1 in Fujimoto and Ranade [10], and consider two matrix equations:

$$M \cdot \begin{bmatrix} X & U \\ V & W \end{bmatrix} = I_X, \text{ and} \tag{1}$$

$$EY = I_S, \tag{2}$$

where $X \in L(S,S)$, $U \in L(T,S)$, $V \in L(S,T)$, and $W \in L(T,T)$, while Y is in $L(S,S)$. We know

$$X = \widehat{E},\ U = \widehat{F},\ V = \widehat{G},\ W = \widehat{H},\ \text{and}\ Y = E^{-1}.$$

Subtracting the second equation (2) from the top part of (1) for domain S, we get

$$M\begin{bmatrix} X-Y & U \\ V & W \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -GY & I_T \end{bmatrix}, \text{ or} \tag{3}$$

$$\begin{bmatrix} X-Y & U \\ V & W \end{bmatrix} = M^{-1}\begin{bmatrix} 0 & 0 \\ -GY & I_T \end{bmatrix} = \begin{bmatrix} \widehat{E} & \widehat{F} \\ \widehat{G} & \widehat{H} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -GY & I_T \end{bmatrix}. \tag{4}$$

Eq.(4) leads to

$$X - Y = -\widehat{F}GY \quad \text{and} \quad V = -\widehat{H}GY.$$

From the latter, we have $GY = -(\widehat{H})^{-1}V = -(\widehat{H})^{-1}\widehat{G}$, which is substituted to the former, yielding $\widehat{E} - E^{-1} = \widehat{F}(\widehat{H})^{-1}\widehat{G}$. □

NB. In the case of the Euclidean space of a finite dimension, $(M^{-1}/\widehat{H}) \equiv \widehat{E} - \widehat{F}(\widehat{H})^{-1}\widehat{G}$ is called the Schur complement of M^{-1} with respect to $\widehat{H}$, and $E^{-1} = \widehat{E} - \widehat{F}(\widehat{H})^{-1}\widehat{G}$ is the Duncan identity.

Corollary 1. If M and E are invertible, then $\widehat{H}$ is invertible.

Proof. In Theorem 1, we assume the existence of inverse for M, E, and $\widehat{H}$. However, if M and $\widehat{H}$ are invertible, we can derive Y as

$$Y = \widehat{E} - \widehat{F}(\widehat{H})^{-1}\widehat{G}.$$

By proceeding backward in the proof of Theorem 1, it is clear that this Y is the right inverse of E. It is also not difficult to show Y is the left inverse of E. Therefore, E is invertible when M and $\widehat{H}$ are invertible. Then, in a dual way, i.e., considering $(M^{-1})^{-1} = M$, we can assert that $\widehat{H}$ is invertible when M and E are invertible..□

Now let us proceed to a more specific case, and take up a linear map A from X into itself, and define

$$M \equiv I - A.$$

This A is also decomposed as

$$A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

We make the following assumptions:

Assumption A1. The linear maps A, A_{11}, and A_{22} are compact (or complete continuous).

Assumption A2. The linear map A is order-preserving (or positive), i.e., if $x \leq y$ for $x, y \in X$, then $Ax \leq Ay$.

Assumption A3. There exists two quasi-interior points d° and x° in X_+ such that $Ax^\circ + d^\circ \leq x^\circ$, i.e., $Mx^\circ \geq d^\circ$.

In Assumption A1, we require three maps A, A_{11}, and A_{22} be compact. However, when a concrete kernel of integral operators is designated, the compactness of A may be sufficient for the other operators to be so. (For the terms from functional analysis, see Yoshida[17] and Schaefer[15].)

We write $A \geq 0$ when the operator A is order-preserving. Note that when $A \geq 0$, $Ax \geq 0$ for any $x \geq 0$.

To begin with, we prove

Lemma 1. Given the assumptions A1-A3, the operators M, E, and H are invertible.

Proof. Consider the operator M, and the solvability of equation $Mx = d$ for $d \in X$. If there exists a positive scalar k such that $d \leq kd^\circ$, we can decompose $d = kd^\circ - (kd^\circ - d)$ with both $kd^\circ \in X_+$ and $(kd^\circ - d) \in X_+$. Thus, we may, without loss of generality, consider the solvability of equation $Mx = d$ only for $d \in X_+$. For $d \in X_+$ such that $d \leq kd^\circ$ with some positive scalar k, we can construct a sequence of vectors in X_+ as

$$\begin{aligned} x^{(1)} &= d \leq kd^\circ \leq kx^\circ, \\ x^{(2)} &= Ax^{(1)} + d \leq A(kd^\circ) + kd^\circ \leq kx^\circ, \\ &\cdots \\ x^{(n+1)} &= Ax^{(n)} + d \leq A(kd^\circ) + kd^\circ \leq kx^\circ, \\ &\cdots \end{aligned}$$

It is clear that $x^{(n)} \leq x^{(n+1)}$ because $x^{(n)} = (A^{(n-1)} + \cdots + A + I) \cdot d$. The sequence $\{x^{(n)}\}_{n \in N}$ is bounded with an upper bound being kx°, and the sequence $\{Ax^{(n)}\}_{n \in N}$ is contained in $\{x^{(n)}\}_{n \in N}$, hence the sequence $\{x^{(n)}\}_{n \in N}$ has a limit $x^* \in X_+$ because A is compact. Since A is continuous, we have $x^* = Ax^* + d$, i.e., $Mx^* = d$. Since by assumption d° is a quasi-interior point, the equation $Mx = d$ can have a solution for any d in a dense set of X. It is well known that this property is enough to show $(I - A)$ is invertible when A is compact. Therefore, $M \equiv I - A$, is shown to be invertible.

Since $E \equiv I_S - A_{11}$ and $H \equiv I_T - A_{22}$, the above proof for M can be applied because A_{11} and A_{22} are also order-preserving, and the assumption A3 is satisfied with x° and d° restricted to S for E, and to T for H. □

Theorem 2. Given the assumptions A1-A3, $\widehat{F}(\widehat{H})^{-1}\widehat{G} \geq 0$.

Proof. Thanks to Corollary 1 and Lemma 1, we know $\widehat{F}(\widehat{H})^{-1}\widehat{G} = \widehat{E} - E^{-1}$ from Theorem 1. The operator $\widehat{E}$ is the operator $(I - A)^{-1} = \sum_{i=0}^{\infty} A^i$ with its domain restricted to S. On the other hand, the operator E^{-1} is $(I_S - A_{11})^{-1} = \sum_{i=0}^{\infty}(A_{11})^i$. Since A is order-preserving, and A_{11} is the operator A with its domain and image restricted to S, it follows that $\widehat{E} - E^{-1} \geq 0$. □

When the set S is made smaller to S', $S' \subsetneqq S$, and $T' \supsetneqq T$, we obtain $(E')^{-1}$ corresponding to this shrinkage of S. We can prove

Theorem 3. Given the assumptions A1-A3, $E^{-1} - (E')^{-1} \geq 0$.

Proof. The proof of Theorem 2 applies, regarding E^{-1} as $\widehat{E}$ and $(E')^{-1}$ as E^{-1}. □

This is an infinite dimensional version of one of what Morishima[13] did in a more detailed way assuming the indecomposability of the material input coefficient matrix A.

4 Remarks

It is to be noted that we do not require the existence of proper interior points in the assumptions, thus allowing L_2 spaces to be dealt with. This is because our operator A is compact, and it is of the form $(I - A)$.

Our theorems may be extended to *nonlinear* operators on Banach spaces as was done in Fujimoto[8] for the n-dimensional Euclidean space. One of the problems is the existence of nonlinear compact operators which are easily verified and economically meaningful.

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