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研究ノート

A Geometrical Essence of Nonsubstitution Theorems

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Abstract. In this note, we give a proposition which shows a geometrical essence of the nonsubstitution theorems. This proposition is concerned with the universality of an optimal solution of linear programs in Hilbert spaces. The essence is that a given linear map covers the positive cone in the target space even when it is restricted to the subspace of the domain which is spanned by the subbasis contained in that optimal solution. For the reader's convenience, we also give a rather extended list of articles on nonsubstitution theorems, though many of them are not cited in the main body.

Keywords: Hilbert Spaces, Linear Programs, Inverse Positivity, Nonsubstitution Theorems

1 Introduction

The purpose of this note is to consider the nonsubstitution theorems in linear economic models in abstract spaces, and extract a geometric essence of nonsubstitution in spite of changes in final demands. We consider a linear program which is to minimize labour input while producing a specified final demand vector. The production is described by a linear map appearing in models of normal activity analysis. It is assumed that we can find an optimal activity vector for a 'strictly positive' final demand vector. The essence obtained is that the given linear map covers the positive cone in the target(final demand commodity) space, even when it is restricted to the subspace of the domain which is spanned by the subbasis contained in that optimal solution. This essence was already recognized in Dasgupta and Sinha[11] and Herrero and Villar[24], and yet by considering the theorem in Hilbert spaces, still more abstract essence can be picked up by use of a linear map restricted to a subspace of orthonormal complement. The proof method comes from Chander[8], and is based on the duality in linear programming problems.

In the next section, we explain notation and state our assumptions. Then in section 3, the main proposition is presented. The final section contains some comments and remarks.

2 Notation and Assumptions

Let X and Y be Hilbert spaces over the real field $\mathbb{R}$, and we assume X has a complete orthonormal basis, $\{\varphi_j\}$. The spaces X and Y have a pointed closed convex cone X_+ and Y_+ respectively formed by vectors x such that $(x, \varphi_j) \geq 0$ for any j, and that Y_+ has a nonempty set, Y_+°, of quasi-interior points. (For quasi-interior points, see Schaefer[43, p.96]) These cones X_+ and Y_+ induce a respective order in X and Y. The symbol M means a given bounded linear map from X into Y, and M^* the adjoint map of M from Y to X, while ℓ and d° are given elements such that $\ell \in X_+$ and $d^\circ \in Y_+^\circ$. We consider the following linear programming problem:

$$\min_x \, (\ell, x) \quad \text{subject to } Mx \geq d^\circ \text{ and } x \in X_+, \tag{P}$$

where $(\ ,\ )$ means the inner-product, and its dual:

$$\max_p \, (d^\circ, p) \text{ subject to } M^*p \leq \ell \text{ and } p \in Y_+. \tag{D}$$

We make the following assumptions.

Assumption 1. The primal problem (P) and its dual (D) have a pair of optimal solutions, x^* and p^*, such that there is no duality gap, i.e., $(\ell, x^*) = (d^\circ, p^*)$.
Assumption 2. $M^*p^* \neq \ell$.

We define the subbasis $\Phi^* \equiv \{\varphi_j \,|\, (\varphi_j, \ell - M^*p^*) = 0\}$, the subspace X^* spanned by this subbasis, and $X_+^* \equiv X^* \cap X_+$. One more important symbol is:

Symbol. $\Pi \equiv \{y \,|\, y = Mx \text{ for } x \in X_+^*\}$.

Thanks to Assumption 1, an optimal solution x^* to the problem (P) is in X_+^*. Since d° is a quasi-interior point, for any $d \in Y$, we can find a sequence of vectors $\{d^{(i)} | i = 1, 2, \dots\}$ such that $\{d^{(i)}\}$ converges to d and, for each $d^{(i)}$, there exists $x^{(i)}$ which satisfies $Mx^{(i)} \geq d^{(i)}$.

3 Main Proposition

We now prove our main proposition which is a nonsubstitution theorem for final demand vectors in Π.

Proposition. Suppose that $d \in \Pi$, and thus $Mx = d$ for $x \in X_+^*$. Then, if $Mz \geq d$ for $z \in X_+$, we have $(\ell, x) \leq (\ell, z)$.

Proof.

From $Mz \geq d$, we get $(p^*, Mz) \geq (p^*, d)$ because $p^* \in Y_+$. On the other hand, from $M^*p \leq \ell$, we obtain $(p^*, Mz) \leq (\ell, z)$ because $z \in X_+$. Therefore,

$$(p^*,\, d) \leq (\ell,\, z). \tag{1}$$

Similarly, from $M^*p \leq \ell$, it follows $(p^*,\, Mx) = (\ell,\, x)$ because $x \in X_+^*$, while we know $(p^*,\, Mx) = (p^*,\, d)$ from $Mx = d$. Thus,

$$(p^*,\, d) = (\ell,\, x). \tag{2}$$

Two eqs. (1) and (2), give the desired result. □

4 Discussions and Remarks

Assumption 1 requires that we should be able to find out an optimal solution, and there should be no duality gap. In the finite dimensional case, this is automatically satisfied so long as we have a feasible vector in the primal problem because the value $(\ell,\, x)$ is nonnegative, thus bounded from below. We may be able to prove that there is no duality gap when a given space X has a countable basis, or an orthonormal basis which may not be countable.

Assumption 2 tells, in the case of a finite dimension, that some processes make losses when prices are p^*, and so, are not used in the optimal solution x^*. And the space X_+^* stands for the space which is spanned by those vectors in the orthonormal basis(processes) which are 'active'(or included) in the optimal x^*. The set Π is the final demand vectors which are producible by using the processes involved in creating the space X_+^* .

Our main proposition simply asserts that a final demand vector in Π can be produced by the processes involved in creating the space X_+^* with the minimum amount of labour among various possible ways of producing that final demand vector without being restricted to the space X_+^* . Note that d is not required to be in Y_+ .

Then, why is our main proposition a geometric essence of the nonsubstitution theorems? This is because the theorems so far made, after all, prove that the set Π covers the positive cone Y_+ . More specifically, suppose that a given linear map M restricted to the space X_+^* is denoted by M^*, and if this M^* is a square matrix in the case of $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, with $n > m$, and that M^* is inverse positive, then we know $\Pi \supset Y_+$. That is, for any nonnegative final demand vector, the same set of processes remain efficient. Finally, when M^* is of the form, $I - A$, where I is the identity matrix and A is a nonnegative matrix, the mere productiveness condition, i.e., the existence of a vector x such that M^*x is in the interior of Y_+ , is enough to render M^* inverse positive, an M-matrix. (See Berman and Plemmons[3].)

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