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研究ノート

Sufficient Conditions for the Weak Hawkins-Simon Property after a Suitable Permutation of Columns

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Abstract. It is shown in Fujimoto-Ranade [5] that an inverse-positive real square matrix has the weak Hawkins-Simon property after a suitable permutation of columns. In a recent paper by Bidard [1], he proved that if the last column of the inverse of a real square matrix is strictly positive, the matrix enjoys the weak Hawkins-Simon property after a suitable permutation of columns. In this note, we present a series of sufficient conditions which bridge these two results, making use of the Gaussian elimination method.

Keywords: Weak Hawkins-Simon condition, Inverse-positivity, Jacobi's determinant identity

1 Introduction

The so-called Hawkins-Simon condition [8] requires that every principal minor be positive, and they showed the condition to be necessary and sufficient for a Z-matrix(a matrix with nonpositive off-diagonal elements) to be inverse-positive. Georgescu-Roegen [12] argued that for a Z-matrix it is sufficient to have only *leading*(upper left corner) principal minors positive, which was also presented in Fiedler and Ptak [4]. Nikaido's two books, [9] and [10], contain a proof based on mathematical induction. Dasgupta [3] gave another proof using an economic interpretation of indirect input.

In this note as in Fujimoto and Ranade [5], the Hawkins-Simon property is defined to be the one which requires that all the *leading* principal minors should be positive, and we shall refer to it as the *weak* Hawkins-Simon property(WHS for short). It has been shown by Fujimoto and Ranade [5] that the WHS property is necessary for a real square matrix to be inverse-positive after a suitable permutation of columns (or rows). Let us call this property the WHSaPC. In a recent paper, Bidard [1] has proved that if the last column of the inverse of a real square matrix is *strictly* positive, the matrix enjoys this WHSaPC. Bidard [1] contains also an interesting observation on some mappings on the real square matrices which preserve this property.

The purpose of this article is to give a series of sufficient conditions for the WHS property, which bridges these two results.

Section 2 explains our notation, then in section 3 we present our theorems and their proofs, finally giving some numerical examples in section 4.

2 Notation

The $(i,\, j)$ element of a matrix $A \in \mathbb{R}^{n\times n}$, $n \geq 2$, is written as a_{ij}, and the i-th element of a vector x as x_i. The j-th column of the matrix A is denoted by $(A)_{*,j}$, while $(A)_{i,*}$ means the i-th row of A. The inequality signs for comparison of two matrices A and $B \in \mathbb{R}^{n\times n}$ are as follows:

$A \geq B$ iff $a_{ij} \geq b_{ij}$;
$A > B$ iff $a_{ij} \geq b_{ij}$ and $A \neq B$;
$A \gg B$ iff $a_{ij} > b_{ij}$.

The same inequality signs are used for the comparison of two vectors.

We use the same notation as in Ouellette[11] and Galántai[7], and let

$$A \equiv \begin{bmatrix} E & F \\ G & H \end{bmatrix} \text{ and } A^{-1} \equiv \begin{bmatrix} \widehat{E} & \widehat{F} \\ \widehat{G} & \widehat{H} \end{bmatrix},$$

where E is in $\mathbb{R}^{p\times p}$, $1 \leq p < n$, and $H \in \mathbb{R}^{q\times q}$, $q = n - p$.

3 Propositions

What Bidard [1] shows is that if the last column of A^{-1} is strictly positive, i.e., $(A^{-1})_{*,n} \gg 0$, then A has the WHSaPC. The main result in Fujimoto and Ranade [5] is that if $A^{-1} > 0$, then A has the WHSaPC. Using the proof method based upon the Gaussian elimination, we can give a series of sufficient conditions, bridging these two results.

We consider the following Condition (P):

(P): There exists a positive integer k $(1 \leq k \leq n)$ and k positive reals, $\alpha_j > 0$ for $j = n - k + 1, \ldots, n$ such that

$$\begin{cases} \text{(i) } \sum_{j=n-k+1}^{n} \alpha_j \cdot (A^{-1})_{*,\,j} \gg 0, \text{ and} \\ \text{(ii) if } k > 1, \text{ then } (A^{-1})_{*,\,j} > 0 \text{ for } (n-k+2) \leq j \leq n. \end{cases}$$

This condition (P) requires that the sum of the last k columns of the inverse A^{-1} with positive weights should form a strictly positive vector, and the last $(k-1)$ columns of A^{-1} is nonnegative and nonzero. Any inverse-positive matrix satisfies this condition with $k = n$ and $\alpha_j = 1$ for $j = 1, \ldots, n$, and the Bidard's case is covered with $k = 1$ and $\alpha_n = 1$. We can now have

Theorem 1. Let A satisfy the condition (P), then the WHS condition is satisfied when a suitable permutation of columns is made.

Proof. We proceed as in the proof of Theorem 3.1 of Fujimoto and Ranade [5]. That is, we consider the linear equation $Ax = b$, and eliminate, step by step, a variable whose coefficient is positive. The existence of such a variable is guaranteed at each step by Condition (P) above. By performing a suitable permutation of columns if necessary, this coefficient can be shown to be positively proportional to a leading principal minor of A.

Because of Condition (P) above, we can choose a vector b such that $b_1 > 0$ and the corresponding solution $x \gg 0$. Thus, there should be at least one positive entry in the first row of A. So, such a column and the first column can be exchanged. We assume the two columns have been permuted so that

$$a_{11} > 0.$$

Now we fix b_1, and in the second step, we divide the first equation of the system $Ax = b$ by a_{11} and subtract this equation side by side from the i-th($i \geq 2$) equation after multiplying this by a_{i1}, to obtain

$$\begin{bmatrix} 1 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ 0 & a_{22} - a_{12}a_{21}/a_{11} & \cdots & a_{2n} - a_{1n}a_{21}/a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} - a_{12}a_{n1}/a_{11} & \cdots & a_{nn} - a_{1n}a_{n1}/a_{11} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1/a_{11} \\ b_2 - b_1 a_{21}/a_{11} \\ \vdots \\ b_n - b_1 a_{n1}/a_{11} \end{bmatrix}.$$

Notice that the $(2, 2)$-entry of the coefficient matrix above is

$$\frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{a_{11}},$$

and the corresponding entry on the RHS is

$$\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{a_{11}}.$$

We continue this type of elimination up to the k-th step, fixing b_{k-1} of the previous step, and having at the (k, k)-position

$$\frac{\begin{vmatrix} a_{11} & \cdots & \cdots & a_{1,k} \\ \vdots & \ddots & & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,k} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1,k-1} \\ \vdots & \ddots & \vdots \\ a_{k-1,1} & \cdots & a_{k-1,k-1} \end{vmatrix}},$$

and the RHS of the k-th equation is given as

$$\frac{\begin{vmatrix} a_{11} & \cdots & a_{1,k-1} & b_1 \\ \vdots & \ddots & & \vdots \\ a_{k,1} & \cdots & a_{k,k-1} & b_k \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1,k-1} \\ \vdots & \ddots & \vdots \\ a_{k-1,1} & \cdots & a_{k-1,k-1} \end{vmatrix}}.$$

The denominator of these equations is known to be positive at the $(k-1)$-th step, and when b_k is large enough, the RHS of the k-th equation becomes positive. By Condition (P), especially thanks to Part (ii), we can again choose a vector $b \gg 0$ such that the equation $Ax = b$ has its corresponding solution $x \gg 0$. Thus, there is at least one positive coefficient in the k-th equation. Again, we assume a suitable permutation has been made so that the (k,k)-position is positive, giving

$$\begin{vmatrix} a_{11} & \cdots & \cdots & a_{1,k} \\ \vdots & \ddots & & \vdots \\ a_{k,1} & \cdots & \cdots & a_{k,k} \end{vmatrix} > 0 \quad \text{for } k = 2, 3, \ldots, n.$$

Therefore, our theorem is proved. □

Bidard [1] offers an interesting observation that three types of maps from $\mathbb{R}^{n\times n}$ to itself keep the WHSaPC property. These are (1) lower triangular matrices with positive diagonal elements, (2) matrices with only one positive entry in each column and each row, and (3) $(P_0 \cdot A^{-1})^t$, where $(\ )^t$ stands for transposition and $P_0 \mathbb{R}^{n\times n}$ is the matrix such that $(P_0)_{i,n-i+1} = 1$ with the remaining entries being all zero. The third group comes out of the Jacobi's determinant identity, i.e., $\left|\widehat{H}\right| \cdot |A| = |E|$. That is, when a matrix A has the WHS property, the matrix $P_0 \cdot A^{-1}$ has also the same property. In words, a sufficient condition on the inverse of A for the WHSaPC, can be transformed to that on the matrix A itself, when the order of columns is reversed and transposed. Thus, we define the following condition (Q):

(Q): There exists a positive integer k $(1 \leq k \leq n)$ and k positive reals, $\alpha_i > 0$ for $i = 1, \ldots, k$ such that

$$\begin{cases} \text{(i) } \sum_{i=1}^{k} \alpha_i \cdot (A)_{i,*} \gg 0, \text{ and} \\ \text{(ii) if } k > 1, \text{ then } (A)_{i,*} > 0 \text{ for } 1 \leq i \leq k-1. \end{cases}$$

Theorem 2. Let A satisfy the condition (Q), then the WHS condition is satisfied when a suitable permutation of columns is made.

Proof. The third group in the observation of Bidard [1] is used together with our Theorem 1. □

Our conditions are, in a sense, still "too much" sufficient because what is required in the k-th step of elimination is the positivity of x_i only for $i = k, \ldots, n$. The Jacobi's determinant identity tells us that when A^{-1} is an M-matrix, A has the WHS property. Bidard [2] contains a necessary and sufficient condition for a real square matrix to have the WHS property. See also Fujimoto, Hisamatsu and Ranade [6] for Schur complements and its relationship to the Le Chatelier-Braun principle.

4 Numerical Examples

Consider the following 3 by 3 matrix and its inverse:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 2 & -1 \\ \frac{1}{2} & \frac{3}{2} & -1 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix}.$$

This matrix satisfies the condition (P) with $k = 2$ and $\alpha_2 = \alpha_3 = 1$. After a permutations columns, the matrix is transformed to

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ \frac{3}{2} & -1 & \frac{1}{2} \end{pmatrix},$$

which has the WHS property.

When the above A is mapped by the third group of Bidard, it becomes

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -1 & -2 \end{pmatrix}.$$

After a permutation of columns, this becomes

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & -1 \end{pmatrix},$$

which surely enjoys the WHS property.

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