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研究ノート

The Weak Hawkins-Simon Property after a Suitable Permutation of Columns: Dual Sufficient Conditions

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Abstract. Fujimoto-Ranade [5] showed that an inverse-positive real square matrix has the weak Hawkins-Simon property after a suitable permutation of columns. In a recent paper by Bidard [1], he proved that if the last column of the inverse of a real square matrix is strictly positive, the matrix enjoys the weak Hawkins-Simon property after a suitable permutation of columns. Then Fujimoto-Ranade [7] gave a series of sufficient conditions which bridge these two sets of conditions. This note represents a dual form of sufficient conditions based upon one of Bidard's observations in [1]. Also, using this, we present a proposition which contains a sufficient condition under which a matrix cannot satisfy the weak Hawkins-Simon property even after any permutation of columns, thus cannot be an inverse-positive matrix.

Keywords: Weak Hawkins-Simon condition, Inverse-positivity, Dual Representation

1 Introduction

In the so-called Hawkins-Simon condition [11], it is required that every principal minor be positive, and it is shown that the condition is necessary and sufficient for a Z-matrix(a matrix with nonpositive off-diagonal elements) to be inverse-positive. Georgescu-Roegen [10] gave an observation that for a Z-matrix it is sufficient to have only *leading*(upper left corner) principal minors positive, which was also proved in Fiedler and Ptak [4]. Nikaido's two books, [15] and [16], contain a proof of this observation based on mathematical induction. Dasgupta [3] gave another proof using an economic interpretation of indirect input, the approach of which comes from the methods in Jeong ([12], [13]) and in Fujita ([8]).

In this note as in Fujimoto and Ranade [5], the Hawkins-Simon property is defined to be the one which requires that all the *leading* principal minors should be positive, and we shall refer to it as the *weak* Hawkins-Simon property(WHS for short). It was shown in Fujimoto and Ranade [5] that the WHS property is necessary for a real square matrix to be inverse-positive after a suitable permutation of columns (or rows). We call this property the WHSaPC. In a recent paper, Bidard [1] has proved that if the last column of the inverse of a real square matrix is *strictly* positive, the matrix enjoys this WHSaPC. Bidard [1] contains also interesting observations on some mappings on the real square matrices which preserve this property. Then, Fujimoto and Ranade [7] gave a series of sufficient conditions which fill the gap between the two results, those in [5] and in [1].

The purpose of this note is to give in a dual form a set of sufficient conditions for the WHSaPC property. A proposition concerning an impossibility for a matrix to have the WHSaPC, is obtained in an easy way through this approach.

Next section 2 explains our notation, then in section 3 represents our theorems and their proofs. In section 4, some numerical examples are given, and the final section 5 contains some remarks.

2 Notation

The $(i,\, j)$ element of a matrix $A \in \mathbb{R}^{n\times n}$, $n \geq 2$, is written as a_{ij}, and the i-th element of a vector x as x_i. The j-th column of the matrix A is denoted by $(A)_{*,j}$, while $(A)_{i,*}$ means the i-th row of A. The inequality signs for comparison of two matrices A and $B \in \mathbb{R}^{n\times n}$ are as follows:

$A \geq B$ iff $a_{ij} \geq b_{ij}$;
$A > B$ iff $a_{ij} \geq b_{ij}$ and $A \neq B$;
$A \gg B$ iff $a_{ij} > b_{ij}$.

The same inequality signs are used for the comparison of two vectors.

We use the same notation as in Ouellette[17] and Galántai[9], and let

$$A \equiv \begin{bmatrix} E & F \\ G & H \end{bmatrix} \text{ and } A^{-1} \equiv \begin{bmatrix} \widehat{E} & \widehat{F} \\ \widehat{G} & \widehat{H} \end{bmatrix},$$

where E is in $\mathbb{R}^{p\times p}$, $1 \leq p < n$, and $H \in \mathbb{R}^{q\times q}$, $q = n - p$.

Some other symbols used are explained in the next section where they appear.

3 Propositions

A sufficient condition Bidard [1] obtained is that if the last column of A^{-1} is strictly positive, i.e., $(A^{-1})_{*,n} \gg 0$, then A has the WHSaPC. The main result in Fujimoto and Ranade [5] is that if $A^{-1} > 0$, then A has the WHSaPC. Fujimoto and Ranade [7] has bridged by a series of sufficient conditions, employing a proof method based upon the Gaussian elimination. That is, we consider the following Condition (TR):

(TR): There exists a positive integer k $(1 \leq k \leq n)$ and k positive reals, $\alpha_j > 0$ for $j = n - k + 1, \ldots, n$ such that

$$\begin{cases} \text{(i) } \sum_{j=n-k+1}^{n} \alpha_j \cdot (A^{-1})_{*,\, j} \gg 0, \text{ and} \\ \text{(ii) if } k > 1, \text{ then } (A^{-1})_{*,\, j} > 0 \text{ for } (n-k+2) \leq j \leq n. \end{cases}$$

This condition (TR) requires that the sum of the last k columns of the inverse A^{-1} with positive weights should form a strictly positive vector, and the last $(k-1)$ columns of A^{-1} is nonnegative and nonzero. Any inverse-positive matrix satisfies this condition with $k = n$ and $\alpha_j = 1$ for $j = 1, \ldots, n$, and the Bidard's case is covered with $k = 1$ and $\alpha_n = 1$. Bidard then wrote to us in an e-mail about a more general sufficient condition.

Theorem 1 (Bidard). Let A satisfy the condition that, in each row of the inverse A^{-1}, the last non-zero entry is positive, then the WHS condition is satisfied when a suitable permutation of columns is made.
Proof. The same proof as that for Theorem 1 of Fujimoto and Ranade [7] applies. The point is that in each step of elimination, we can find a positive solution x^* even when the entry b_i at that step is large. □

Bidard [1] offers an interesting observation that three types of maps from $\mathbb{R}^{n\times n}$ to itself keep the WHSaPC property. These are (1) lower triangular matrices with positive diagonal elements, (2) matrices with only one positive entry in each column and each row, and (3) $P_0 \cdot (A^{-1})^t$, where $(\ )^t$ stands for transposition and $P_0 \in \mathbb{R}^{n\times n}$ is the matrix such that $(P_0)_{i,n-i+1} = 1$ with the remaining entries being all zero. The third group comes out of the Jacobi's determinant identity, i.e., $\left|\widehat{H}\right| \cdot |A| = |E|$. That is, when a matrix A has the WHS property, the matrix $P_0 \cdot A^{-1} \cdot P_0$ (as well as $P_0 \cdot (A^{-1})^t \cdot P_0$) has also the same property. In words, a sufficient condition on the inverse of A for the WHSaPC, can be transformed to that on the matrix A itself, when the order of columns is reversed and transposed. We wish to represent this fact more explicitly, using Bidard's method.

In what follows, P denotes a general permutation matrix in $\mathbb{R}^{n\times n}$, thus $P^t = P^{-1}$. The above P_0 is a special permutation matrix such that $P_0^t = P_0 = P_0^{-1}$ or $P_0^2 = I$, where I is the identity matrix. The set of matrices which satisfy the weak Hawkins-Simon condition is written as $\mathbb{H}$.

Proposition 1. Suppose that when a matrix A^{-1} has a property (P), A has the WHSaPC. Then, if $A^t \cdot P_0$ has the property (P), A has the WHSaPC.
Proof. Since $A^t \cdot P_0$ has the property (P), we have $(A^t \cdot P_0)^{-1} = HP$ for some $H \in \mathbb{H}$. Thus, $A^t \cdot P_0 = P^t \cdot H^{-1}$. From this, it follows $A^t = P^t \cdot H^{-1} \cdot P_0$. Hence,

$$A = P_0 \cdot (H^{-1})^t \cdot P = (P_0 \cdot (H^{-1})^t \cdot P_0) \cdot (P_0 \cdot P).$$

This shows what is desired. □

Proposition 2. Suppose that when a matrix A has a property (Q), A has the WHSaPC. Then, if $P_0 \cdot (A^{-1})^t$ has the property (Q), A has the WHSaPC.
Proof. Since $P_0 \cdot (A^{-1})^t$. has the property (Q), we have $P_0 \cdot (A^{-1})^t = HP$ for some $H \in \mathbb{H}$. Thus, $(A^{-1})^t = P_0 \cdot H \cdot P$. From this, it follows $A^{-1} = P^t \cdot H^t \cdot P_0$. Hence,

$$A = P_0 \cdot (H^t)^{-1} \cdot P = (P_0 \cdot (H^{-1})^t \cdot P_0) \cdot (P_0 \cdot P).$$

This proves the proposition. □

Proposition 1 tells us that when a property (P) for inverse matrices is sufficient for the WHSaPC, if a matrix after transposition and reverse ordering of rows has that property, then it has the WHSaPC. On the other hand, Proposition 2 tells us that when a property (Q) for a matrix is sufficient for the WHSaPC, if an inverse matrix after transposition and reverse ordering of columns has that property, then it has the WHSaPC.

Therefore, we may state

Theorem 2. Let A satisfy the condition that, in each column of A, the last non-zero entry is positive, then the WHS condition is satisfied when a suitable permutation of columns is made.

Proof. We apply Proposition 1 to Theorem 1. □

Another consequence is

Theorem 3. If the last column of the inverse A^{-1} is non-positive, i.e., $(A^{-1})_{*,n} < 0$, then A cannot satisfy the WHSaPC.

Proof. It is clear that if there is a matrix A which has the WHSaPC with the stated property realized by its inverse A^{-1}, we can consider a property (P) with those particular values of A^{-1}. Then, by virtue of Proposition 1, the matrix with its first row being non-positive (and certainly non-zero) has the WHSaPC, which is a contradiction. □

Here it seems desirable to give the following.

Theorem 4. If a regular matrix contains a non-zero non-positive row (or column), it cannot be inverse-positive.

Proof. Without loss of generality, we can assume the first row of a given matrix is non-zero and non-positive. This cannot satisfy the WHSaPC, and so cannot be positive-inverse thanks to Theorem 3.1 in Fujimoto and Ranade [5]. In the case of columns, we can apply our argument to the transposed matrix. □

This Theorem 4 may be compared with a result by Johnson [14], which asserts that if a regular matrix in $\mathbb{R}^{n\times n}$ contains a strictly positive column (or row), it cannot be inverse-positive.

4 Numerical Examples

Consider the following 3 by 3 matrix and its inverse:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This matrix satisfies the condition stated in Theorem 1. After a permutation of columns, the matrix is transformed to

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

which has the WHS property, verifying Theorem 1.

When the above A^{-1} is transposed and its ordering of rows is reversed, we have

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix},$$

which has the WHS property. This confirms Proposition 1 and Theorem 2.

Next consider

$$A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}.$$

Then the original matrix is

$$A = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & -1 \end{pmatrix},$$

thus verifying Theorem 3, and also Theorem 4, replacing A and A^{-1}.

There are matrices which satisfy the condition by Johnson [14], that there is at least one strict positive column (or row), and has the WHSaPC property. For example,

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

On the other hand, there can be matrices which satisfy the Johnson's condition, and yet does not have the WHSaPC. For example,

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}.$$

Its inverse is

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$$

5 Remarks

Remark 1. The results obtained above are related to the Le Chatelier-Samuelson principle in Leontief models. See Fujimoto, Hisamatsu and Ranade [6]. This principle has much to do with Schur complements. For the latter, refer to [17] and [9].
Remark 2. One can find a necessary and sufficient condition for a matrix to have the WHSaPC in [2]. His condition does not, however, seem to be easy to verify and to obtain concrete sufficient conditions.

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